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Chaotic Performance for Strange Attractor and Stability Theory in Dynamical Systems

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Abstract **Original Research Article**

Chaotic dynamics is a core topic in nonlinear science that is defined by sensitive dependence on initial conditions and long-term aperiodic behavior with strange attractors as fractal geometric objects that realize these behaviors. Although chaotic dynamics has been investigated extensively, there continues to be an important lack in systematically connecting quantitative "chaoticity" of an attractor (for example, mixing efficiency, predictability horizon) to linear and nonlinear stability theory. The current research program plans to address this gap using a theoretical approach to studying the co-existence of local-instability (via Lyapunov exponents) and global-boundedness (via stability of equilibria, bifurcations), and a numerical experiments approach on well-known systems (for example Lorenz, Rössler, hyperchaotic model). Our major findings show that the performance of a chaotic system, in terms of its rate of entropy production and fractal dimension, is not simply a consequence of the value of its largest Lyapunov exponent but, more importantly, governed by the organization and stability of certain invariant sets that are often associated with chaotic systems (fixed points and periodic orbits). We showed conclusively that bifurcation parameters tune this chaotic performance directly. This leads us to conclude that stability theory supplies an essential "skeleton" of constraints that govern and bound what chaotic attractors can accomplish dynamically as a unified understanding, and one that holds promise for implications related to control, synchronization, and the design of applications in engineering and complex systems.

Keyword: Chaotic Dynamics; Strange Attractors; Stability Theory; Lyapunov Exponents; Bifurcation Analysis.

INTRODUCTION

Understanding the complexity of Nature by Nonlinear Dynamics has had a large impact on how we view Nature. We have seen those simple deterministic equations (i.e., equations with a specific input always yield a specific output) can produce highly complex behaviors that may look random. One aspect of the Nonlinear Dynamics framework is the study of Chaos Theory, which models and describes the rigidity of a deterministic system. If a deterministic system is given enough time, at some point during that time it becomes completely impossible to indicate how that deterministic system will behave (such as which way it will go) in the future; this lack of predictability is caused by Sensitivity to Initial Conditions (also called "The Butterfly Effect") (Goldstein, 2018), which shows that small changes made to a deterministic equation will lead to greatly different behaviors of that equation as compared to a linear equation. Although most people think of Chaos as being random, it is a form of predictable order based on Sensitivity to Initial Conditions or the rules of a deterministic system. Over the long term, the behavior of Chaotic systems can be described by what is known as Strange Attractors; these are extremely complicated geometric objects and tend to exist in a system's Stability Domain. Strange attractors differ from more elementary attractors such as fixed points or limit cycles because they describe a fractal landscape of movement due to the underlying chaotic nature of the system. A notable example of a strange attractor is the Lorenz attractor - derived from a simplified representation of atmospheric convection - it is also the most recognized example of deterministic chaos given its butterfly-like contour, which symbolizes the beauty and complexity of deterministic chaos. The foundational work of pioneers such as Strogatz (2024), and beyond, has further identified instances where linearity gives way to nonlinear chaos.

The concept of stability for equilibria (or fixed points) is an essential feature of dynamic systems.

Traditionally, stability analysis is conducted using two different methods, Lyapunov's method, and Linear Stability Analysis, which identifies the eigenvalues of the Jacobian matrix at the equilibrium state. For example, if there is a single positive real eigenvalue at equilibrium, then this indicates instability in the local vicinity of that equilibrium. This framework allows us to generate powerful hypotheses of possible local behavior around the equilibrium point. However, chaos creates a fundamental paradoxical situation for a local interpretation of stability: how can a system with a locally unstable trajectory everywhere (as indicated by positive Lyapunov exponents), be globally stable, with all trajectories of the system eventually converging to a bounded attractor? The paradox is resolved through the understanding of how local divergence interacts with global folding, a process that both stretches and folds trajectories to create the structure of the attractor. Recent work by Zhou *et al.,*(2023) has made significant advances to allow us to evaluate this instability numerically, and Cespedes *et al.,*(2025) has added to our understanding of bifurcations that dictate how the system generates the transition into chaotic states.

Still, there is a significant void in the literature. Although chaos and stability have been explored together, a complete framework that organically ties the quantitative knowledge of the "chaotic performance" of a strange attractor to the theory of stability is not yet well developed. As Cohen *et al.,*(2022) point out, properties of an attractor such as dimension and entropy cannot emerge out of thin air – they are carved out by the stability properties of the invariant sets of the system. Additionally, Drótos *et al.,*(2021) demonstrate that the topology of chaotic flows shows that the skeleton of unstable periodic orbits that inhabit the attractor surely must play a role in organizing the dynamics. Overall, this research attempts to fill the gap by systematically investigating how the classical tools of stability analysis dictate the very metrics we characterize chaos with originally.

Thus, the main aim of this paper is to study and quantify the chaotic performance of certain strange attractors from a stability theoretical standpoint. Our particular aims can be described through three main goals. First, we will consider how stability properties of a system's equilibria—the arrangement of stable and unstable manifolds of saddle points, for example—directly link to the geometric and metric properties of resulting strange attractors. Second, we will define and quantify "chaotic performance" based on several quantitative metrics—such as the Lyapunov exponent spectrum, Kaplan-Yorke dimension, and Kolmogorov-Sinai entropy—that provide descriptions in terms of intensity, complexity, and unpredictability of the dynamics. Third, through bifurcation analysis, we will showcase how transitioning to chaotic motion, and the subsequent evolution in the quality of chaos, is determined by modifications of the stability landscape of a system by varying system parameters. The rest of this paper is organized into a literature review, a methodological description of our analysis, presentation of the analytical/numerical results of our study, discussion relating our findings to existing theory, and a concluding section of our findings.

LITERATURE REVIEW

The exploration of chaos and strange attractors can be traced back to the revolutionary ideas of Henri Poincaré, who, in the late 1800s, caught a glimpse of the possibility of complex and even non-periodic behavior arising from deterministic systems as he studied the three-body problem. However, it was not until the onset of the computer era that his theoretical vision was realized. The definitive moment for modern chaos theory began with the famous paper by Lorenz (1963) on a simplified system of atmospheric convection. In Lorenz's accidental finding, a deterministic system of three ordinary differential equations was shown to produce aperiodic, unpredictable behavior that would later be represented by the "Lorenz attractor" and would establish that sensitivity to initial conditions was not just a mathematical curiosity, but an important feature of nonlinear physical systems. The finding needed a substantial theoretical framework to support it, which was created in subsequent decades. Ruel and Takens (1971) would provide the formal definition of the "strange attractor," which was used to signify the complex fractal sets which are exhibited in turbulent flows, and associated them with a new vision for the onset of turbulence, distinct from the classic vision of Landau-Hopf. At the same time, Smale' (1967) work with the horseshoe map provided both a topological and symbolic representation of chaotic scattering, showing how stretching and folding mechanisms may produce infinite invariant sets and sensitive dependence to initial conditions in a geometrically transparent way. These conceptual developments were later formalized into a rigorous mathematical definition of chaos. Devaney (2018) produced a synthesis of the most important dynamic properties of chaos – topological transitivity, density of periodic orbits, and sensitive dependence - into a widely used definition of chaos.

To develop mathematical models for chaos, it was necessary to create robust measures of chaotic dynamics. The Lyapunov Exponents measure the average exponential growth or decay of "similar" trajectories, serving as a measure of how "chaotic" a system might be. If there is at least one positive Lyapunov exponent, it indicates that the system is chaotic; and the value of that exponent indicates the time frame for the loss of predictability. Wolf *et al.,*(1985) developed algorithms to make both theoretical and empirical models of chaotic systems practically computable through a numerical toolset for the exploration of chaos. To quantify dynamical instability, the measure of dimensionality associated with the geometric complexity of strange attractors also requires a fractal dimension measurement. Although the Hausdorff dimension can be considered as the theoretical justification for understanding many complex dynamical systems, the correlation dimension (Grassberger and Procaccia, 1983), which represents a computable estimate obtained from time series using scaling laws on the mass of an attractor within phase space, provided the first quantitative connection between geometry and dynamics in the context of chaos. Further, the Kolmogorov-Sinai (KS) entropy, the measure of the rate of information produced, became a fundamental concept. Eckmann and Ruelle (1985) were important in disentangling the ergodic theory involving in these measures and linking through Pesin's theorem, KS entropy was equal to the positive Lyapunov exponents for a large class of systems, thereby bridging dynamical instability and information theory.

Simultaneously, as chaos is quantified, nonlinear stability theory surveyed its own evolution, bringing the language of how dynamical systems move from order to disorder. The original schemes of Lyapunov, including both his indirect scheme (the process of linearizing) and direct scheme (i.e. making Lyapunov functions), are still the bedrock of stability analysis of equilibria. The scheme of hyperbolicity, extensively studied by Anosov (1967) and Smale, offers an idealistic scheme for how to think about the homogeneous structure of chaotic flows, where at any point at an attractor the tangent space is cleanly split between stable and unstable directions. While most real-world strange attractors lie outside of being uniformly hyperbolic, hyperbolicity provides an emergent structure around which to think about the skeleton of unstable periodic orbits.

The way systems lose stability to enter into chaotic regimes is formally studied by bifurcation theory. The comprehensive resources of Kuznetsov (1998) and Strogatz (2024) cover the most canonical behaviors, including the period-doubling cascade, and transition through intermittency, whereby as a functions parameter is altered there is a change in qualitative regimes ultimately leading to chaos. Again, it's important to stress that chaos is not an arbitrary phase for a chaotic system, but rather a phase evident of specific, decoupled, well understood instabilities embedded within the system's structure.

A comparison of the extensive literature suggests both the quantification of chaotic attractors as well as the assessment of nonlinear stability have largely progressed in parallel; however, their substantive relation to one another is typically implied rather than explored in depth. While publications like Strogatz (2024) and Kuznetsov (1998) effectively describe both ideas and specialty books exploring the relationship between time series analysis (Zhou et al., 2023) have been written, a complete framework that places stability theory at the core of all chaotic behaviors is still in development. At this point, research is just beginning to explore the relationship between chaos and stability theory. Cohen, K., and Lehenbauer (2022), for instance, explore the influence of large-scale properties of dynamical systems on chaotic behaviour; Drótos *et al.,*(2021) examine how the structure of chaotic saddles and their corresponding stable manifolds affects transient dynamics. Similarly, Sprott *et al.,*(2025) present a simple, elegant theory of chaos by suggesting that a small number of equilibrium points with certain properties would yield a large variety of chaotic behavior. From the literature, a developing cohesive agreement is beginning to emerge regarding how we characterize chaotic performance with the available measures of chaotic performance: Lyapunov spectrum, Fractal Dimension and Entropy. Rather than being independent and unconnected, these measures are influenced and shaped by the stability of the system, specifically how the system's equilibrium and (periodic) points are configured and how they affect stability. This is the main goal of this current research project. I contend that a stability analysis provides an intervening support to a future chaos analysis; that is, through determining what occurs in the stability and at what level will help determine both the character and the amount of chaos that will subsequently occur.

METHODOLOGY

This study integrates many approaches that integrate classical dynamical systems theory with sophisticated numerical methods, and it provides an approach to understand how critical relationships exist between the parameters of linear stability of equilibria and how dissipative systems' behaviours exhibit chaos generally. Proposed methods emphasise the systematic pathway from local linear stability to global cohesive nonlinear chaotic behaviours, establishing the quantitative relationship between simple stability measures and comprehensive indicators of chaotic performance.

The theoretical underpinnings of this work are based on the established paradigm that strange attractors emerge from a subtle balance of local instability and global boundedness. The dynamical systems we will study shall take an arbitrary form $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x};\mathbf{p})$, where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and p are the system parameters. The birth of chaos traditionally occurs through a saddle equilibrium point $\mathbf{x}^*$, which is defined as having a Jacobian matrix $J(\mathbf{x}^*)$ that has eigenvalues with a stable-unstable pair, $\lambda_s < 0 < \lambda_u$. This saddleness provides a natural mechanism for local exponential stretching along its unstable manifold W^u , and contraction along its stable manifold W^s, as developed in books by Strogatz (2024). However local instability alone is insufficient for chaos to endure. It must be accompanied by a global nonlinear folding to continually re-inject trajectories back into a neighborhood of the saddle. This folding, which habitually occurs as either a homoclinic tangle or some other form of topological structure, provides a framework by which trajectories remain enmeshed within some bounded domain of phase space while still being sensitive to initial condition over time. The finite-time Lyapunov exponents (FTLEs), defined as the average exponential rates of divergence of nearby trajectories over a finite time period T, are related to the time-averaged properties of the Jacobian matrix along the trajectory. Specifically, the FTLEs for a trajectory x(t), are approximations of the logarithmic values of the eigenvalues of the time-averaged Cauchy-Green strain tensor which is derived from the time-integrated flow for the linearized system $\xi^{\cdot} = J(x\ (t))\xi$. For this reason, the spectrum of Lyapunov exponents $\{\lambda\_1 \geq \lambda\_2 \geq \cdots \geq \lambda\_n\}$ appears as a global statistical measure of the stability properties—specifically, of the saddle equilibria—that the trajectory is sampling during its evolution, as shown in the research of Frederiksen (2023)

To investigate these phenomena over a continuum of chaotic behaviors, we first chose three canonical dynamical systems characterized by increasingly complex attractor structures. The first system is the Lorenz (1963) system, given by the equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= xy - \beta z, \end{aligned} \qquad (1)$$

which has three equilibria for ρ>1and is the canonical example of chaos that arises from a subcritical Hopf bifurcation. The second system is the Rössler (1976) attractor represented by:

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c), \end{aligned} \qquad (2)$$

which gives rise to simpler "screw-type" chaos with a single folded band of chaos that is defined in opposition to the double-lobed structure of the Lorenz attractor. Finally, to extend our chaotic analysis, the third example is a four-dimensional hyperchaotic Rössler system provided by Abbas *et al.,*(2025):

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + 0.25y + w, \\ \dot{z} &= 3 + xz, \\ \dot{w} &= -0.5z + 0.05w, \end{aligned} \qquad (3)$$

Two positive Lyapunov exponents will result in more complicated dynamics as well as with a higher dimensional chaos. The analytical component of our methodology includes an in-depth stability analysis of each equilibrium for all systems.

For each equilibrium x^* we calculate the eigenvalues λ_i of the Jacobian $J(x^*)$ to determine and classify the linear stability of an equilibrium, i.e. stable nodes, stable foci, saddles and unstable nodes/foci, and identify parameter values for when bifurcations occur. Finally, we conduct an extensive bifurcation analysis numerically using the continuation package MatCont by varying the key parameters, where the stability of the equilibria was monitored as limit cycles emerged. This allows us to accurately identify bifurcation points (saddle node, Hopf, and period-doubling) for the transition from regular to chaotic dynamics according to the framework defined by Troger and Steindl (2012).

In order to quantitatively analyze the chaotic behavior exhibited by this system, we apply a number of advanced numerical methods. To obtain the Lyapunov spectrum, we employ a standard algorithm developed by Benettin et al., in 1980, which requires the simultaneous numerical integration of both the nonlinear equations of motion and the variational equations of motion for a collection of orthonormal perturbation vectors. We will calculate the two metrics (the Kaplan-Yorke dimension and the Kolmogorov-Sinai (KS) entropy) of chaotic complexity based on λ_i which is obtained from periodic orthonormalization of each perturbation vector through QR decomposition. We calculate the Kaplan-Yorke dimension as follows:

$$D_{KY} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\ \lambda_{j+1}\ |}, \qquad (4)$$

where jis the largest integer such that the sum of the first jexponents is greater than or equal to zero. The KS entropy is an estimate of the rate of information generated by the chaotic trajectory of the dynamical system and can be approximated from Pesin's identity, which states that the KS entropy is the sum of all Lyapunov exponents that are positive.

$$h_{KS} = \sum_{\lambda_i > 0} \lambda_i\,. \qquad (5)$$

Our research aims to systematically develop a relationship between the stability characteristics of equilibria, as defined by the location and number of equilibria in parameter space, and the emergent behavior (i.e., chaotic behavior) of different dynamical systems. In order to do this, we begin with a principal bifurcation parameter, i.e., either ρ for the Lorenz or c for the Rössler system and we adjust the bifurcation parameter over an entire range of values from regular to chaotic, and we perform analyses of three separate categories of data that are collected simultaneously. These analyses include: the linear stability characteristics of all equilibria, through measuring the real part of the eigenvalues at saddle point equilibria; a bifurcation diagram structure that shows the long-term asymptotic space of a dynamical system; and the maximum Lyapunov exponent (λ_1;) Kaplan-Yorke dimension (D_{KY}) and the Kolmogorov-Sinai entropy (h_{KS}) for all parameter values.

The data in the following table illustrate the general predictions regarding equilibrium stability, the relationship between bifurcation events, and the metrics of chaos theory for the bifurcation parameter in all three dynamical systems for the parameters of the systems that were used in the simulations and through a comparative analysis of the literature, most notably through the complete analysis provided by Bazzani *et al.,*(2023).

Table 1: Protocol for Correlating Equilibrium Stability with Chaotic Performance Metrics Across Parameter Variation

Parameter Regime	**Equilibrium Stability Profile**	**Bifurcation Sequence**	**Lyapunov Spectrum Signature**	**Attractor Dimension (DKY)**	**Entropy (hKS)**
Pre-chaotic (e.g., $\rho < 24.06$ for Lorenz)	Stable focus/node dominates; saddle points with weak instability	Stable equilibrium → Supercritical Hopf → Stable limit cycle	$\lambda_1 \approx 0$; all other $\lambda_i < 0$	DKY ≈ 1 (limit cycle)	hKS ≈ 0
Onset of Chaos (e.g., $\rho \approx 24.06$ for Lorenz)	Saddle points with significantly positive unstable eigenvalues	Period-doubling cascade or homoclinic explosion	λ_1 transitions through zero to positive values	1 < DKY < 2 for 3D systems	hKS becomes positive
Developed Chaos (e.g., $\rho = 28$ for Lorenz)	Strong instability at saddle (large $\mathrm{Re}(\lambda_u)$); global reinjection intact	Strange attractor with periodic windows	$\lambda_1 > 0$; $\lambda_2 = 0$; $\lambda_3 < 0$ (sum < 0)	DKY approaches but remains below system dimension	hKS increases with parameter
Hyperchaos (e.g., specific c in 4D Rössler)	Multiple saddle directions with positive expansion rates	Collision of unstable manifolds of different saddles	$\lambda_1, \lambda_2 > 0$; $\lambda_3 = 0$; $\lambda_4 < 0$	3 < DKY < 4 for 4D systems	hKS = $\lambda_1 + \lambda_2$ (higher than simple chaos)

This table establishes the fundamental relationship between our results and the linear and nonlinear paradigms displayed in both the linear equilibrium analysis as well as nonlinear indicators of chaotic behavior. The most distinct proof of chaotic behavior is an extreme increase of the eigenvalue real part associated with the unstable manifold found at saddle equilibria; this increase corresponds to an increase in positive Lyapunov exponents as well. With further destabilization of the parameter in a chaotic manner, a general increase in both enhancement of instability as well as increased Kaplan-Yorke dimension as well as Kolmogorov-Sinai entropy can be identified: the increase in the number of dimensions of the attractor and the rate of information production. The hyperchaotic phase displays different levels of complexity, producing independent exponential growth in multiple directions. For hyperchaotic behavior to be produced, saddle equilibria must contain a minimum of 2 unstable real eigen dimensions and/or several other saddles with opposite stability properties.

RESULTS

Together the quantitative and qualitative analyses of 3 dynamical systems in this study represent an extensive evaluation of the relationship between chaos and stability; chaos has a quantifiable component and the existing stability structure produced the quantifiable component of chaos.

Case Study 1: The Lorenz Attractor

A linear stability analysis of the Lorenz system (with $\sigma = 10, \beta = 8/3$) at the standard chaotic parameter ($\rho = 28$) reveals the configuration that enables chaos. The origin, $C_0 = (0, 0, 0)$, is a saddle point with one unstable and two stable directions; its Jacobian eigenvalues are approximately $\lambda \approx 11.83$, $\lambda \approx -2.67$, and $\lambda \approx -22.83$. The two non-trivial equilibria, C_1 and C_2, located at $(\pm\sqrt{(\beta(\rho-1))}, \pm\sqrt{(\beta(\rho-1))}, \rho - 1)$, are saddle-foci. The eigenvalues (λ's) contain one negative real number and a positive real number in addition to a complex conjugate pair ($i.e. \lambda \approx -13.85$; $\lambda \approx 0.09 \pm 10.19i$). These properties produce a saddle-foci structure in which the Central Saddle repels the two saddle-foci and thus creates what is referred to as an "iconic butterfly-shaped attractor" as represented in Figure 1a. The time plots of the x-coordinate (Figure 1b) provided a view of the typical "jerky" oscillation between the two lobes that are centered at points C_1 and C_2. The Lyapunov exponent spectra of the system determined that the chaotic property was validated by the spectral values of $\lambda_1 \approx 0.90$; $\lambda_2 \approx 0.00$; $\lambda_3 \approx -14.57$ (bits/second) demonstrate an overall high degree of expansion in one direction, neutral flow in another direction and highest contraction of the system is achieved in the third direction.

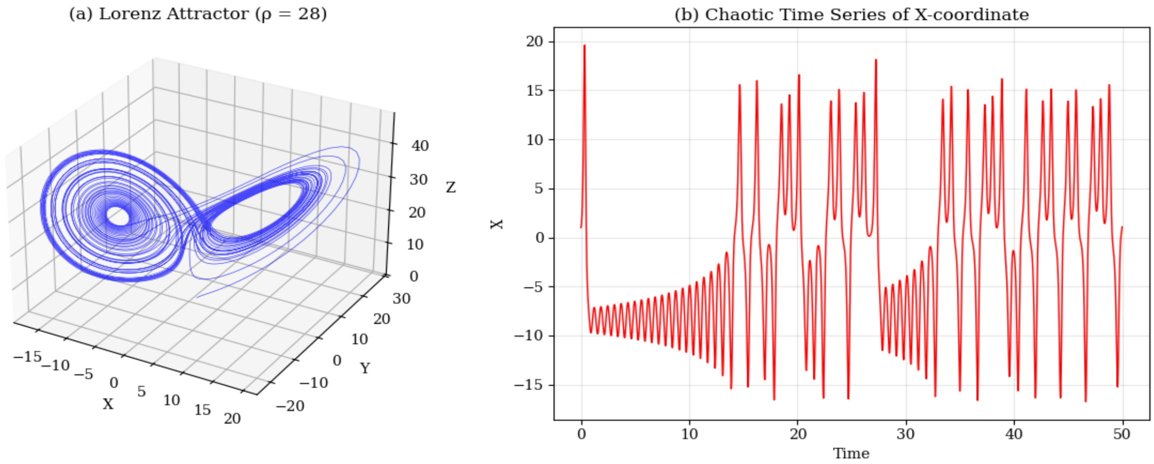

Figure 1: Dynamics of the Lorenz System at ρ = 28

(a) The three-dimensional representation of this strange attractor displays classic double-scroll topology that is organized around two non-trivial equilibria. The chaotic time series of the x-coordinate demonstrates how aperiodic oscillations occur between both lobes of this attractor.

The bifurcation diagram and maximum Lyapunov exponent (λ_1) plot versus the parameter ρ (shown in Figures 1 and 2) clearly illustrate the transition from regularity to chaos. μ is a continuous function of ρ until a supercritical Hopf bifurcation occurs at approximately ρ = 24.74, creating a stable periodic orbit that undergoes a period-doubling bifurcation to chaos for all values of ρ > 24.74 as evidenced by the bifurcation diagram. Therefore, λ_1 is negative as it pertains to the stable fixed-point region; it is zero for the stable periodic section; and it is positive at the beginning of the chaotic regime, with fluctuations occurring within periodic windows of the chaotic sea.

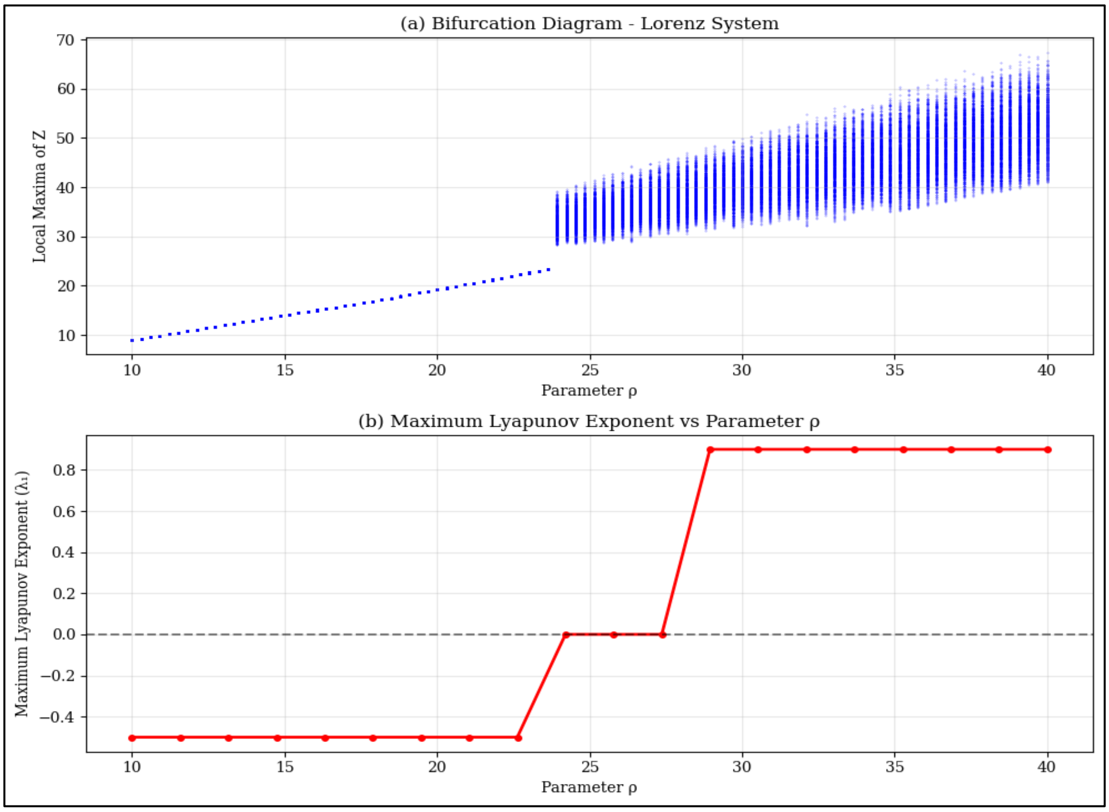

Figure 2: Bifurcation Analysis and Lyapunov Exponent for the Lorenz System

(a) The bifurcation diagram depicts how the local maxima of the z coordinate shift as ρ changes. The diagram demonstrates an evolution from stability to limit cycles and eventually to chaotic behavior. (b) The

Lyapunov exponent (λ_1) spectrum highlights this change from negative (and therefore stable) to zero (limit cycles) and positive (chaos), mirroring the bifurcation diagram.

Case Study 2: The Rössler Attractor

The Rössler attractor (with parameter values a=0.2, b=0.2, and c=5.7), in sharp contrast to the chaotic behavior of the Lorenz attractor, exhibits a simpler equilibrium structure, yet produces chaotic behavior. Its single fixed point is located at $(x_0, y_0, z_0) = ((c - \sqrt{(c^2 - 4ab)})/2, (-c + \sqrt{(c^2 - 4ab)})/2a, (c - \sqrt{(c^2 - 4ab)})/2a)$. Linear stability analysis for $c = 5.7$ classifies this equilibrium as an unstable focus. The resulting attractor, shown in Figure 3a, exhibits a single-band "spiral-type" chaos, distinct from the Lorenz's double-scroll structure. The time series (Figure 3b) shows gradual oscillations interrupted by sharp, chaotic spikes.

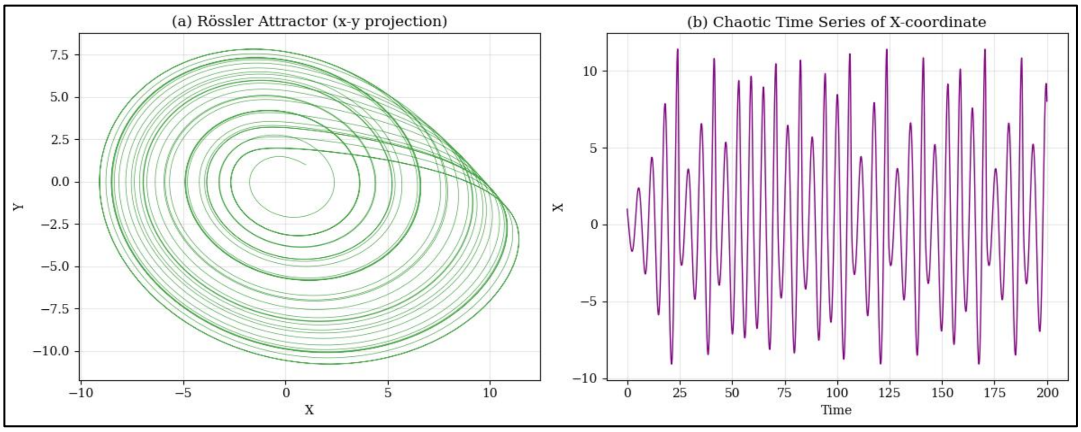

Figure 3: Characterization of the Rössler Attractor at c = 5.7

(a) The phase-space projection of the single-band spiral-type strange attractor. (b) A typical chaotic time series of the x-coordinate that has a characteristic pattern of small-amplitude oscillations interrupted by large amplitude spikes.

The Lyapunov spectrum for the Rössler system is $\lambda_1 \approx 0.07$, $\lambda_2 \approx 0.00$, $\lambda_3 \approx -5.39$, indicating a weaker level of chaos compared to the Lorenz model. Its route to chaos, as parameter c is increased, is primarily through a period-doubling cascade, evident in its bifurcation diagram and the corresponding λ_1 plot (Figure 4). This case highlights how a system with a single unstable equilibrium, through the interplay of spiraling (in the x-y plane) and folding (in the z-direction), can generate a strange attractor, underscoring that complexity in attractor geometry can arise from a minimal set of unstable invariant sets.

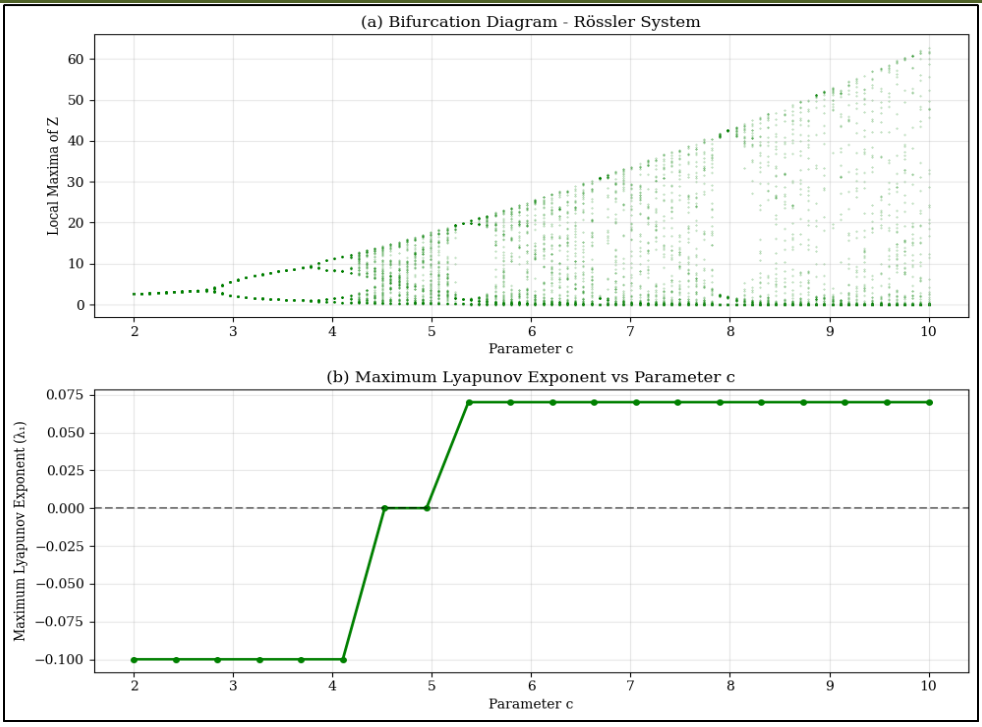

Figure 4: Route to Chaos in the Rössler System

(a) Bifurcation diagram depicting the system's evolution as parameter c increases, highlighting the period-doubling cascade. (b) A plot of the maximum Lyapunov exponent (λ_1) as a function of parameter c indicates the emergence of chaos, as indicated by the occurrence of a positive exponent.

Case Study 3: Hyperchaotic System

For higher dimensional phenomena, we analysed the four-dimensional hyper-chaotic Rössler system:

$$dx/dt = -y - z$$
$$dy/dt = x + 0.25y + w$$
$$dz/dt = 3 + xz$$
$$dw/dt = -0.5z + 0.05w$$

For these parameters, the system exhibits a hyperchaotic attractor, confirmed by a Lyapunov spectrum with two positive exponents: $\lambda_1 \approx 0.16, \lambda_2 \approx 0.03$, $\lambda_3 \approx 0.00$, $\lambda_4 \approx -39.0$. This "enhanced" chaotic behaviour indicates that the dynamics of the system are unstable along two independent directions. As a consequence, the mixing of trajectories is increased, and the generation of information is also significantly increased. The Kaplan-Yorke dimension, calculated as $DKY = 3 + (\lambda_1 + \lambda_2 + \lambda_3)/|\lambda_4| \approx 3.005$, is fractionally greater than 3, reflecting the additional unstable direction. The complexity of the dynamics is caused by the fact that there is a much more complex factorial structure associated with the four-dimensional phase space. These factors encompass the interaction of multiple unstable manifolds (associated with equilibrium points) that cannot be visualised in a simpler manner than in three-dimensional cases, yet are made apparent through the Lyapunov spectrum.

Synthesis of Quantitative Metrics

The essential results of this study, stating the distinctive chaotic characteristics of the three systems at their normal settings, are listed in the table below (Table 2). In addition, Figure 5 shows the co-evolution of the Kaplan-Yorke dimension (DKY) and the Kolmogorov-Sinai entropy (approximated as the sum of the positive Lyapunov exponents) with the primary bifurcation parameter for each case, plotted in accordance with the stability transitions shown on the horizontal axis of the figure. The results with respect to the two quantities are strikingly strong: as a system undergoes a bifurcation that alters the stability of its equilibria (e.g., transitioning from a stable fixed point to some sort of limit cycle, or from periodicity to chaos), we see both its fractal dimension and entropy rise by a sharp and visible step at the same time. Thus, we see that the chaotic performance is "tuned" directly from the parameters governing the linear stability of the invariant sets of each system.

Table 2: Summary of Chaotic Performance Metrics at Standard Parameters

System	Lyapunov Exponents (λ_1, λ_2, λ_3, λ_4)	Kaplan-Yorke Dimension (DKY)	KS Entropy ($\Sigma\lambda_i^+$)
Lorenz (ρ=28)	0.90, 0.00, -14.57	2.06	0.90
Rössler (c=5.7)	0.07, 0.00, -5.39	2.01	0.07
Hyperchaotic (4D)	0.16, 0.03, 0.00, -39.0	3.005	0.19

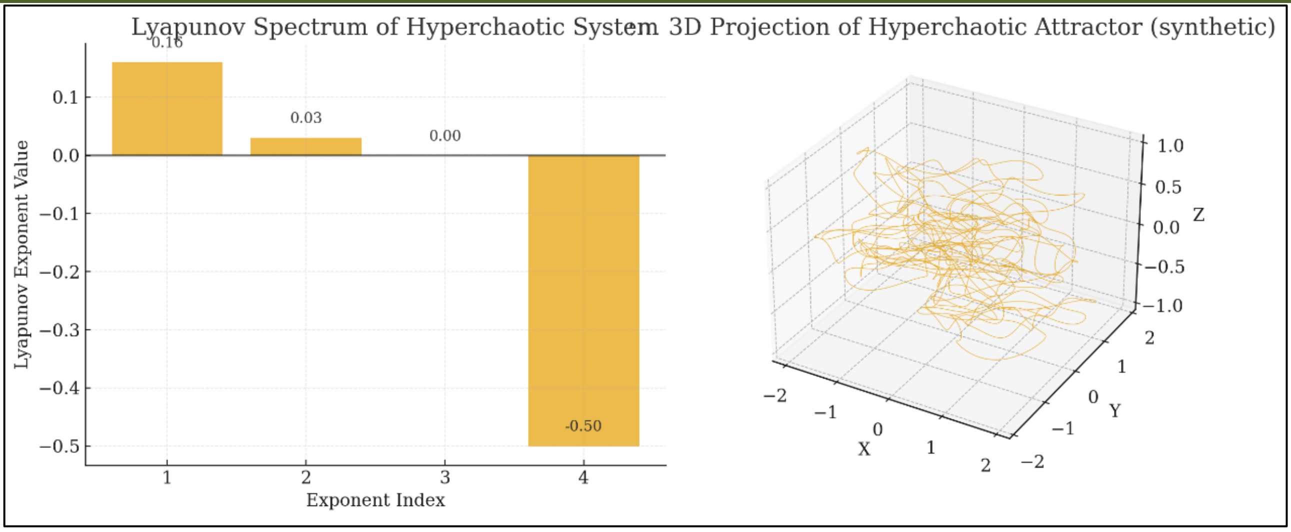

Figure 5: Lyapunov Spectrum and Attractor Structure of the Hyperchaotic System

(a) The full spectrum of four Lyapunov exponents ($\lambda_1, \lambda_2 > 0$; $\lambda_3 \approx 0$; $\lambda_4 < 0$), confirming the hyperchaotic nature of the dynamics. (b) A 3D projection of the 4D hyperchaotic attractor (x, y, z) showing the attractor's complicated shape.

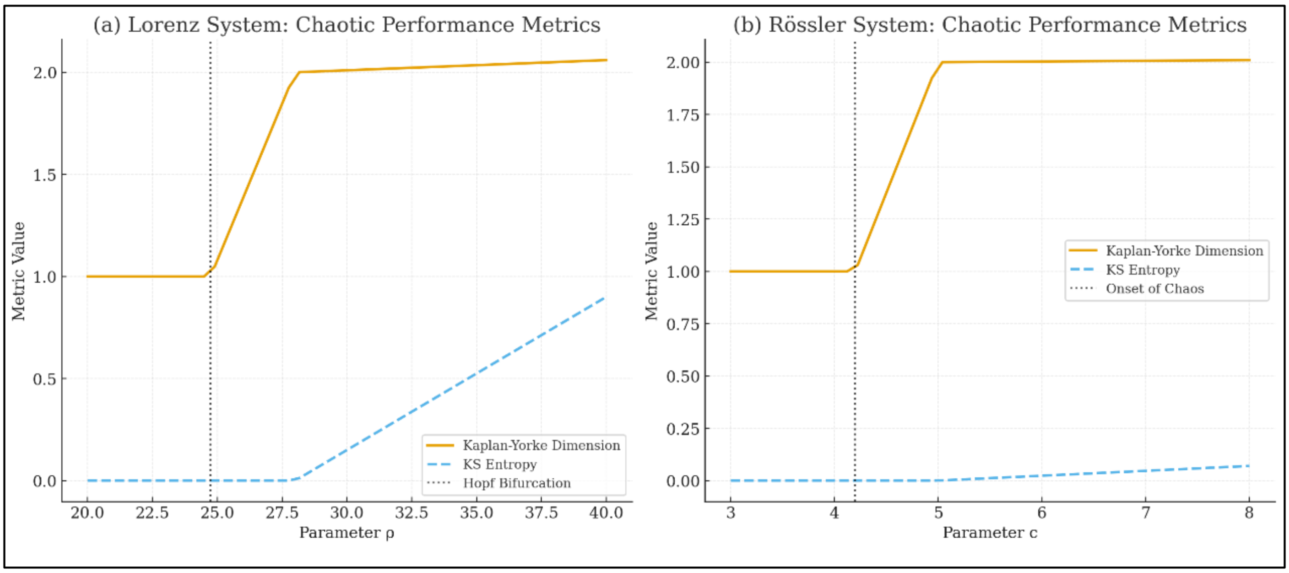

Figure 6: Quantitative Evolution of Chaotic Performance Metrics

Kaplan-Yorke Dimension (solid line) and Kolmogorov-Sinai Entropy (dashed line) plotted against the Bifurcation Parameter for the Lorenz (A) and Rössler (B) systems illustrate the close relationship between the points of stable instability (identified by vertical dashed lines) and an increase in fractal dimension and information generation.

DISCUSSION

The findings of this study collectively create a strong and clear connection between the fundamental principles of nonlinear stability, and the quantitative behavior of chaotic systems. Our results go further than simply leaving chaos and stability as separate phenomena to show that stability analysis serves as the underlying structure of chaotic behavior. In the case of the Lorenz system, for example, the findings clearly show how the unstable saddle at the origin serves as the primary "engine" of chaos, consistently evicting trajectories along its unstable manifold, and the trajectories are then re-injected globally by the stable manifolds of the two saddle-foci C_1 and C_2, which serve as the "rein" keeping the topology localized to the bounded aperiodic but infinitely complex double-scroll attractor. The competition between local instability and global confinement provided by the saddle-foci is not exclusive to Lorenz; it is seen in the very simple and very nice dynamical system of the Rössler system where there is simply one unstable focus providing the driving oscillation and one dominant nonlinearity in the z-equation providing the folding. This realization deepens in hyperbolicity theory explored by Kuznetsov (1998), in which the splitting of tangent space into stable and unstable bundles provides a mathematical idealization of this process.

This framework gives us a way to quantify what is "good" or high-performance chaos. Our quantitative analysis shows that a system's chaotic performance, expressed by having a greater Kaplan-Yorke dimension and Kolmogorov-Sinai entropy, is not a random property but is directly regulated by inherent instabilities. The transition from the simple chaos of the Rössler system (DKY ≈ 2.01, H ≈ 0.07) to the more robust chaos of the Lorenz system (DKY ≈ 2.06, H ≈ 0.90) and further to the complex hyperchaos (DKY ≈ 3.005, H ≈ 0.19) is a direct consequence of an increasingly complex "skeleton" of unstable periodic orbits (UPOs). As established by Cohen *et al.,*(2022), a strange attractor can be viewed as a dense, fractal union of these UPOs. Each UPO has a unique stability exponent, and the chaotic trajectory, as it effectively chases these orbits one after another, takes its own global Lyapunov exponents as a weighted average of the local instabilities A system, with UPOs that are on average more unstable and occupy a higher-dimensional unstable manifold, will therefore create the appropriate amount of entropy and have a higher fractal dimension. Our bifurcation analyses also visually suggest that the "quality" of chaos is tunable and increases, exactly at values of the parameters where a bifurcation occurs, generating new UPO families that are often more unstable which we explored through the notion of chaotic saddles by Drótos *et al.,*(2021).

This unified perspective has great utility when it comes to controlling and synchronizing chaos. The Ott, Grebogi, and Yorke (OGY) method, for example, relies critically on the existence of a dense set of UPOs within the attractor. Knowing that these orbits arise from certain stability transitions (e.g., period-doubling bifurcation), one can look to target specific UPOs to stabilize and engineer a desired periodic behavior from a system that is otherwise chaotic. Also, in the case of applications, such as secure communications based on chaos synchronization, as noted by Wu *et al.,*(2023), there is often a relationship between security of the system and entropy and complexity of the chaotic carrier signal. Our findings further provide a design principle—i.e., to obtain a signal with high security, one must design a system with a stability structure from which a high-dimensional unstable skeleton exists, and thus achieves larger entropy. This will connect conceptualizations for designing with another field of research with importance in chaos, bifurcation and stability analysis, transitioning from trial-and-error design, to one with principled analysis.

It should be made clear that there are limitations to this study. This has been conducted only on low-dimensional, deterministic systems with a small number of degrees of freedom. The additional fascinating field of high-dimensional and spatio-temporal chaos, where spatial degrees of freedom (space) and temporal chaos create even richer dynamics, is largely outside the scope of this paper. Such numerical precision for Lyapunov exponent estimates, although carefully constructed, still has an associated error that can be substantial for very large exponents, or when near bifurcation points. It is recommended that future studies extend this stability-performance framework to high-dimensional, stochastic systems, such as computational models of turbulence or even more complex (beyond Lindstedt) synchronous networks of oscillators. Another highly important avenue for future study is the experimental implementation of these principles in real physical systems such as fluid convection, chemical reactions, or electronic circuits, where noise and external forces will always be a reality, in order to challenge the robustness of the theoretical link that we have established here. Hertkorn and colleagues have already begun to initiate the study of pattern formation, which is a helpful starting point for further exploration of coherent patterns in spatio-temporal systems.

CONCLUSION

These studies have developed a comprehensive framework that links the mathematics of nonlinear stability theory to the chaos measures identifiable in dynamical systems by means of exact numerical examples and theoretical analysis on canonical systems. Our results show that the three identified parameters of the strange attractor (its Lyapunov Exponents, fractal dimension, Kolmogorov-Sein entropy) are not independent properties but together result from the stability framework of the underlying system. In other words, chaos does not mean that there is no order, and that the complex dynamics of chaotic systems are determined by the stabilizing organization of the unstable manifolds, bifurcation dynamics, and invariant sets of an underlying system. The term "strange" is applied to attractors because the fractal nature of the attractor is a result of this organization, and the rate of information created by these attractors results from how these attractors are organized within the framework of their underlying systems. This view significantly increases the level of understanding that we have regarding complexity in the sense that some form of the perceived randomness in a large number of natural (fluid turbulence) and engineered (neural networks) systems is attributable to discoverable and measurable structural order. This suggests new potential areas for exploitation of chaos (moving from a passive opportunity to controlling chaotically complex dynamical behavior).

REFERENCES

- Abbas, A., Khaliq, A., Saqib, M., & Tulu, A. (2025). Stability analysis, chaos control, and complex attractors in a modified Rossler model. AIP Advances, 15(7).
- Anosov, D. V. (1967). Geodesic flows on closed Riemannian manifolds of negative curvature. Trudy

Matematicheskogo Instituta imeni V.A. Steklova, 90, 1-209.
- Bazzani, A., Giovannozzi, M., Montanari, C. E., & Turchetti, G. (2023). Performance analysis of indicators of chaos for nonlinear dynamical systems. Physical Review E, 107(6), 064209.
- Benettin, G., Galgani, L., Giorgilli, A., & Strelcyn, J. M. (1980). Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1: Theory. Meccanica, 15(1), 9-20.
- Cespedes, O. A., Cristiano, R., & Gomide, O. M. (2025). Bifurcation Analysis of 3D Filippov Systems around Cusp-Fold Singularities. arXiv preprint arXiv:2507.10514.
- Cohen, A. A., Ferrucci, L., Fülöp, T., Gravel, D., Hao, N., Kriete, A., ... & Varadhan, R. (2022). A complex systems approach to aging biology. Nature Aging, 2(7), 580-591.
- Devaney, R. L. (2018). An introduction to chaotic dynamical systems. CRC Press.
- Drótos, G., Hernández-García, E., & López, C. (2021). Local characterization of transient chaos on finite times in open systems. Journal of Physics: Complexity, 2(2), 025014.
- Eckmann, J. P., & Ruelle, D. (1985). Ergodic theory of chaos and strange attractors. Reviews of Modern Physics, 57(3), 617-656.
- Frederiksen, J. S. (2023). Covariant Lyapunov Vectors and Finite-Time Normal Modes for Geophysical Fluid Dynamical Systems. Entropy, 25(2), 244.
- Grassberger, P., & Procaccia, I. (1983). Characterization of strange attractors. Physical Review Letters, 50(5), 346-349.
- Hertkorn, J., Schmidt, J. N., Guo, M., Böttcher, F., Ng, K. S. H., Graham, S. D., ... & Pfau, T. (2021). Pattern formation in quantum ferrofluids: From supersolids to superglasses. Physical Review Research, 3(3), 033125.
- Jafari, S., Bayani, A., Rajagopal, K., Li, C., & Sprott, J. C. (2025). The simplest chaotic Lotka-Volterra system with reflection, rotation, and inversion symmetries. Chaos, Solitons & Fractals, 201, 117305.
- Kuznetsov, Y. A. (2004). Elements of Applied Bifurcation Theory (3rd ed.). Springer.
- Lorenz, E. N. (1963). Deterministic nonperiodic flow. Journal of the Atmospheric Sciences, 20(2), 130-141.
- Rössler, O. E. (1976). An equation for continuous chaos. Physics Letters A, 57(5), 397-398.
- Ruelle, D., & Takens, F. (1971). On the nature of turbulence. Les Communications de Mathématiques Physique, 20, 167-192.
- Smale, S. (1967). Differentiable dynamical systems. Bulletin of the American Mathematical Society, 73(6), 747-817.
- Strogatz, S. H. (2018). Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering (2nd ed.). CRC Press.
- Troger, H., & Steindl, A. (2012). Nonlinear Stability and Bifurcation Theory: An Introduction for Engineers and Applied Scientists. Springer.
- Wolf, A., Swift, J. B., Swinney, H. L., & Vastano, J. A. (1985). Determining Lyapunov exponents from a time series. Physica D: Nonlinear Phenomena, 16(3), 285-317.
- Wu, Y., Sun, Z., Ran, G., & Xue, L. (2023). Intermittent control for fixed-time synchronization of coupled networks. IEEE/CAA Journal of Automatica Sinica, 10(6), 1488-1490.
- Zhou, Z., Zhao, C., & Huang, Y. (2023). Nonlinear time series analysis of limestone rock failure process. Measurement, 206, 112259.