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Series and Exponentially-Fitted Three-Points Optimized Hybrid Volterra Integral Equation of the Second Kind for General Third Order Ordinary Differential Equations

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Abstract **Original Research Article**

This study presents series-based and exponentially-fitted three-point optimized hybrid numerical methods for the direct resolution of general third-order ordinary differential equations. The approaches are formulated by transforming the governing problems into second-kind Volterra integral equations, thus obviating the necessity for reduction to analogous first-order systems. Power series and exponential fitting techniques are incorporated as basis functions to improve accuracy and stability, particularly for stiff problems. Theoretical analysis establishes the order of accuracy, local truncation error, consistency, and zero-stability of the schemes, while the regions of absolute stability are also investigated. Numerical studies on specific benchmark issues indicate that the proposed methods attain superior accuracy and enhanced computational efficiency relative to traditional approaches. The results validate the efficacy and robustness of the optimized hybrid Volterra integral framework for addressing general third-order ordinary differential equations.

Keywords: Third-order ordinary differential equations; Volterra integral equation of the second kind; optimized hybrid methods; exponential fitting; numerical stability.

1. INTRODUCTION

The advancement of numerical techniques for higher-order ordinary differential equations has progressed markedly in recent decades. Early methods relied heavily on classical finite difference schemes and Runge–Kutta methods applied after reducing higher-order equations to first-order systems which were discussed in Anake [2011], Gholamtabar and Parandin [2014], Akinfenwa *et al.,* [2018]. However, these techniques often suffer from inefficiency and stability issues when applied to stiff. Traditional numerical approaches for solving third-order ODEs typically involve reducing the equation to an equivalent system of first-order equations before applying standard numerical schemes. Although effective, this approach increases computational cost, memory requirements, and accumulated truncation errors. Consequently, direct numerical methods that approximate the solution of higher-order ODEs without reduction have gained significant attention in recent years. The second kind of Volterra Integral Equation (VIE) is of form:

$$y(x)=f(x)+\int_{\alpha(x)}^{\beta(x)} k(x,s)y(s)\,ds, \tag{1}$$

In this paper, we formulate (1.1) into third-order Initial Value Problem (IVP) to a hybrid Volterra Integral Equation (VIE) of the form

$$y'''(x) = f'''(x) + \int_{\alpha(x)}^{\beta(x)} K\left(x, s, y(s) y'(s), y''(s)\right) ds, \quad x \in [x_0, X] \tag{2}$$

It is evident that resolving (1.2) is synonymous with addressing the initial-value difficulties of third-order Ordinary Differential Equations (ODEs).

$$y'''(x) = f'''(x) + \varphi\left(x, y(x), y'(x), y''(x)\right), \quad y\left(x_0\right) = f\left(x_0\right), \quad y'\left(x_0\right) = f'\left(x_0\right), \quad y''\left(x_0\right) = f''\left(x_0\right), \tag{3}$$

Consequently, the resolution of the integral equation (2) and the initial-value problem (3) can be achieved using a singular technique. Transforming (2) into a system of first-order equations prior to employing an approximate strategy for issue resolution frequently results in increased processing expense. A approach for solving higher-order ordinary differential equations is the direct use of the Predictor-Corrector method, as extensively addressed by Kayode and Adeyeye [2013], Adesanya *et al.,* [2008], and Awoyemi and Idowu [2005] To overcome these limitations, direct numerical methods for higher-order ODEs were introduced. Block and hybrid methods gained prominence due to their ability to generate multiple solution values simultaneously. Researchers such as Adesanya *et al.,* [2014] and Fatunla [1988] demonstrated that hybrid methods offer improved stability and accuracy when compared with conventional single-step methods. Among these approaches, hybrid methods based on Volterra integral equations of the second kind have proven to be powerful tools due to their inherent stability and ability to provide continuous approximations which were discussed in Raymond *et al.,* [2023], Raymond and Kyagya [2020]. Furthermore, the incorporation of optimized hybrid schemes with multiple collocation points enhances accuracy and convergence. In this study, series and exponentially-fitted three-point optimized hybrid methods are developed within the Volterra integral equation framework to address the numerical challenges associated with general third-order ordinary differential equations, particularly those exhibiting stiff. Despite these advances, limited attention has been given to three-point optimized hybrid Volterra integral equation methods that combine power series and exponential fitting for general third-order ODEs. This research seeks to fill this gap by developing robust, accurate, and stable three-point optimized hybrid schemes within the Volterra integral equation framework.

2. METHODS

New Two-Step Hybrid Optimized Volterra Integral Equation of the second kind with three- points of optimization

To address the resolution of (2), we propose utilizing a combination of power series and an exponentially fitted function as the approximate solution, with the algorithm structured accordingly.

$$\varphi'''(x) = \phi'''(x) + \sum_{j=0}^{s} \mu_i x^j + \sum_{j=t}^{q} \psi_j e^{x^j} \tag{4}$$

Be the approximate solution of (2) where μ_i and ψ_i are the coefficients to be determined.

Let the approximate power series and exponentially fitted function be of the form

$$\sum_{i=0}^{k} \tau_i \varphi_{n+i} = \sum_{i=0}^{k} \tau_i \phi_{n+i} + h^3 \sum_{i=0}^{s} \varsigma_i (t) \mu_{n+i} + h^3 \sum_{j=i}^{r} \psi_j (t) e^{x^j} \tag{5}$$

$$\sum_{i=0}^{k} \tau_i \left(\varphi'''_{n+i} - \phi'''_{n+i} \right) = h^3 \left(\sum_{j=0}^{s} \varsigma_j (t) \mu_{n+j} + \sum_{j=i}^{r} \psi_j (t) e^{x^j} \right) \tag{6}$$

where $\tau_i, \varsigma_j . \psi_j, (i=0,1,\cdots,m;\ j=i,\cdots,k)$ are constant coefficient of (6) to be determined which is considered as the solution to the volterra integral equation of the second.

To formulate this strategy, three off-grid sites will be introduced which are u , $\frac{1}{2}$ and v placed between x_n and x_{n+1} respectively. For the two-step method, the value of our k shall be two, therefore we shall optimize u and v .

From (6), we obtain

$$\varphi'''(x) = \phi'''(x) + \sum_{j=0}^{3} \mu_i x^j + \sum_{j=1}^{5} \psi_j e^{x^j} \tag{7}$$

With first and second and third derivative of equation (7) we obtain

$$\varphi'(x)-\phi'(x) = \sum_{j=0}^{3} jx^{j-1}\mu_j(x) + \sum_{j=1}^{5} jx^{j-1}\psi_j(x)e^{x^j} \tag{8}$$

$$\varphi''(x)-\phi''(x) = \sum_{j=0}^{3} j(j-1)x^{j-2}\mu_j(x) + \sum_{j=1}^{5} \psi_j\, j\left[(j-1)x^{j-2}+jx^{2(j-2)}\right]e^{x^j} \tag{9}$$

$$\varphi'''(x)-\phi'''(x) = \sum_{j=0}^{3} j(j-1)(j-2)x^{j-3}\mu_j + \sum_{j=1}^{5} jx^{j-3}\psi_j(x)e^{x^j}\left(j^2x^{2j}-3j-3jx^j+3j^2x^j+j^2+2\right) \tag{10}$$

Interpolating (7) - (9) at the points $x_n = 0$ and collocating (10) at all points $t_{n+m} = t_n + qh, m = \{0, u, \frac{1}{2}, v, 1, 2\}$. Equation (1.4) result to system of nonlinear equation of the form

$$MD = V \tag{11}$$

$$\begin{bmatrix}
12 & 12x_n & 9x_n^2 & 7x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 & \frac{1}{840}x_n^7 & \frac{1}{6720}x_n^8 \\
0 & 12 & 18x_n & 21x_n^2 & x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 & \frac{1}{840}x_n^7 \\
0 & 0 & 18 & 42x_n & 3x_n^2 & 3x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 \\
0 & 0 & 0 & 42 & 6x_n & 3x_n^2 & x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 \\
0 & 0 & 0 & 42 & 6(x_n+uh) & 3(x_n+uh)^2 & (x_n+uh)^3 & \frac{1}{4}(x_n+uh)^4 & \frac{1}{20}(x_n+uh)^5 \\
0 & 0 & 0 & 42 & 6(x_n+\frac{1}{2}h) & 3(x_n+\frac{1}{2}h)^2 & (x_n+\frac{1}{2}h)^3 & \frac{1}{4}(x_n+\frac{1}{2}h)^4 & \frac{1}{20}(x_n+\frac{1}{2}h)^5 \\
0 & 0 & 0 & 42 & 6(x_n+vh) & 3(x_n+vh)^2 & (x_n+vh)^3 & \frac{1}{4}(x_n+vh)^4 & \frac{1}{20}(x_n+vh)^5 \\
0 & 0 & 0 & 42 & 6(x_n+h) & 3(x_n+h)^2 & (x_n+h)^3 & \frac{1}{4}(x_n+h)^4 & \frac{1}{20}(x_n+h)^5 \\
0 & 0 & 0 & 42 & 6(x_n+2h) & 3(x_n+2h)^2 & (x_n+2h)^3 & \frac{1}{4}(x_n+2h)^4 & \frac{1}{20}(x_n+2h)^5
\end{bmatrix}
\begin{bmatrix} \mu_0 \\ \mu_0' \\ \mu_0'' \\ \psi_0 \\ \psi_u \\ \psi_{\frac{1}{2}} \\ \psi_v \\ \psi_1 \\ \psi_2 \end{bmatrix}
=
\begin{bmatrix} (\varphi_0-\phi_0) \\ (\varphi_0'-\phi_0') \\ (\varphi_0''-\phi_0'') \\ g_n \\ g_{n+u} \\ g_{n+\frac{1}{2}} \\ g_{n+v} \\ g_{n+1} \\ g_{n+2} \end{bmatrix}$$

Using Gaussian elimination method on (11) gives the coefficients of μ_j, $\psi_j, j = 0(1)8$. These values are then substituted into (4) to give the implicit continuous optimized hybrid Volterra integral equation of the second kind scheme of the form;

$$\begin{aligned} p\left((\varphi_n+\xi h)-(\phi_n+\xi h)\right) &= \mu_0(\varphi_n-\phi_n)+h\mu_1(\varphi'_n-\phi'_n)+h^2\mu_2(\varphi''_n-\phi''_n) \\ &\quad + h^3\left[\psi_0 g_n+\psi_u g_{n+u}+\psi_{\frac{1}{2}}g_{n+\frac{1}{2}}+\psi_v g_{n+v}+\psi_1 g_{n+1}+\psi_2 g_{n+2}\right] \end{aligned} \tag{12}$$

Differentiating (12) once and twice, we obtain

$$\begin{aligned} hp'\left((\varphi_n+qh)-(\phi_n+qh)\right) &= h\mu_0(\varphi'_n-\phi'_n)+h^2\mu_1(\varphi''_n-\phi''_n) \\ &\quad + h^3\left[\psi_0 g_n+\psi_u g_{n+u}+\psi_{\frac{1}{2}}g_{n+\frac{1}{2}}+\psi_v g_{n+v}+\psi_1 g_{n+1}+\psi_2 g_{n+2}\right] \end{aligned} \tag{13}$$

$$h^2p''\left((\varphi_n+qh)-(\phi_n+qh)\right) = h^2\mu_0(\varphi''_n-\phi''_n)+h^3\left[\psi_0 g_n+\psi_u g_{n+u}+\psi_{\frac{1}{2}}g_{n+\frac{1}{2}}+\psi_v g_{n+v}+\psi_1 g_{n+1}+\psi_2 g_{n+2}\right] \tag{14}$$

Substituting $\xi = 1$ in (12), we obtain an approximate volterra integral equation to the solution of (3) at the points t_{n+1} which yield

$$p\left(\left(\varphi_n + h\right) - \left(\phi_n + h\right)\right) = \left(\varphi_n - \phi_n\right) + h\mu_1\left(\varphi'_n - \phi'_n\right) + \tfrac{1}{2}h^2\mu_2\left(\varphi''_n - \phi''_n\right)$$
$$+h^3\left[\begin{array}{l}\frac{1}{1680}\left(\frac{-13u-13v+119uv+2}{uv}\right)g_n + \frac{1}{840}\left(\frac{13v-2}{u(u-1)(u-2)(2u-1)(u-v)}\right)g_{n+u} \\ +\frac{2}{315}\left(\frac{-22u-22v+70uv+9}{(2v-1)(2u-1)}\right)g_{n+\frac{1}{2}} - \frac{1}{840}\left(\frac{13u-2}{v(v-1)(v-2)(2v-1)(u-v)}\right)g_{n+v} \\ -\frac{1}{840}\left(\frac{-u-v+14uv-1}{(v-1)(u-1)}\right)g_{n+1} + \frac{1}{5040}\left(\frac{-u-v+7uv}{(v-2)(u-2)}\right)g_{n+2}\end{array}\right] \quad (15)$$

Also Substituting $\xi = 1$ in (13) we obtain an approximate volterra integral equation to the solution of (3) at the points t'_{n+1} which yield

$$hp'\left(\left(\varphi_n + h\right) - \left(\phi_n + h\right)\right) = h\left(\varphi'_n - \phi'_n\right) + h^2\mu_2\left(\varphi''_n - \phi''_n\right) + h^3$$
$$+h^3\left[\begin{array}{l}\frac{1}{840}\left(\frac{-14u-14v+133uv+1}{uv}\right)g_n + \frac{1}{420}\left(\frac{14v-1}{u(u-1)(u-2)(2u-1)(u-v)}\right)g_{n+u} \\ +\frac{8}{315}\left(\frac{-21u-21v+56uv+10}{(2v-1)(2u-1)}\right)g_{n+\frac{1}{2}} - \frac{1}{420}\left(\frac{14u-1}{v(v-1)(v-2)(2v-1)(u-v)}\right)g_{n+v} \\ -\frac{1}{420}\left(\frac{7u+7v+7uv-8}{(v-1)(u-1)}\right)g_{n+1} + \frac{1}{2520}\left(\frac{7uv-1}{(v-2)(u-2)}\right)g_{n+2}\end{array}\right]$$

Finally, substituting $\xi = 1$ in (14) we obtain an approximate volterra integral equation to the solution of (3) at the points t'_{n+1} which yield

$$h^2p''\left(\left(\varphi_n + h\right) - \left(\phi_n + h\right)\right) = h^2\left(\varphi''_n - \phi''_n\right)$$
$$+h^3\left[\begin{array}{l}\frac{1}{840}\left(\frac{-u-v+20uv-1}{uv}\right)g_n + \frac{1}{60}\left(\frac{v+1}{u(u-1)(u-2)(2u-1)(u-v)}\right)g_{n+u} \\ +\frac{8}{45}\left(\frac{-7u-7v+15uv+4}{(2v-1)(2u-1)}\right)g_{n+\frac{1}{2}} - \frac{1}{60}\left(\frac{u+1}{v(v-1)(v-2)(2v-1)(u-v)}\right)g_{n+v} \\ +\frac{1}{60}\left(\frac{-11u-11v+10uv+10}{(v-1)(u-1)}\right)g_{n+1} + \frac{1}{360}\left(\frac{u+v+1}{(v-2)(u-2)}\right)g_{n+2}\end{array}\right] \quad (17)$$

Expanding (15) - (17) using Taylor series around the points t_n we obtain

$$\begin{bmatrix} \sum_{j=0}^{2}\frac{(h^3)}{j!}(\varphi_n-\phi_n)^j-(\varphi_n-\phi_n)-h(\varphi'_n-\phi'_n)-\frac{1}{2}h^2(\varphi''_n-\phi''_n)-\sum_{j=0}^{2}\frac{h^{j+2}}{j!}+h^3\left[\begin{bmatrix}\frac{1}{1680}\left(\frac{-13u-13v+119uv+2}{uv}\right)g_n+\frac{1}{840}\left(\frac{13v-2}{u(u-1)(u-2)(2u-1)(u-v)}\right)g_{n+u}\\ +\frac{2}{315}\left(\frac{-22u-22v+70uv+9}{(2v-1)(2u-1)}\right)g_{n+\frac{1}{2}}-\frac{1}{840}\left(\frac{13u-2}{v(v-1)(v-2)(2v-1)(u-v)}\right)g_{n+v}\\ -\frac{1}{840}\left(\frac{-u-v+14uv-1}{(v-1)(u-1)}\right)g_{n+1}+\frac{1}{5040}\left(\frac{-u-v+7uv}{(v-2)(u-2)}\right)g_{n+2}\end{bmatrix}\right] \\ \sum_{j=0}^{2}\frac{(h^3)}{j!}(\varphi_n-\phi_n)^j-h(\varphi'_n-\phi'_n)-\frac{1}{2}h^2(\varphi''_n-\phi''_n)-\sum_{j=0}^{2}\frac{h^{j+2}}{j!}+h^3+h^3\begin{bmatrix}\frac{1}{840}\left(\frac{-14u-14v+133uv+1}{uv}\right)g_n+\frac{1}{420}\left(\frac{14v-1}{u(u-1)(u-2)(2u-1)(u-v)}\right)g_{n+u}\\ +\frac{8}{315}\left(\frac{-21u-21v+56uv+10}{(2v-1)(2u-1)}\right)g_{n+\frac{1}{2}}-\frac{1}{420}\left(\frac{14u-1}{v(v-1)(v-2)(2v-1)(u-v)}\right)g_{n+v}\\ -\frac{1}{420}\left(\frac{7u+7v+7uv-8}{(v-1)(u-1)}\right)g_{n+1}+\frac{1}{2520}\left(\frac{7uv-1}{(v-2)(u-2)}\right)g_{n+2}\end{bmatrix} \\ \sum_{j=0}^{2}\frac{(h^3)}{j!}(\varphi_n-\phi_n)^j-h^2(\varphi''_n-\phi''_n)-\sum_{j=0}^{2}\frac{h^{j+2}}{j!}++h^3\begin{bmatrix}\frac{1}{840}\left(\frac{-u-v+20uv-1}{uv}\right)g_n+\frac{1}{60}\left(\frac{v+1}{u(u-1)(u-2)(2u-1)(u-v)}\right)g_{n+u}\\ +\frac{8}{45}\left(\frac{-7u-7v+15uv+4}{(2v-1)(2u-1)}\right)g_{n+\frac{1}{2}}-\frac{1}{60}\left(\frac{u+1}{v(v-1)(v-2)(2v-1)(u-v)}\right)g_{n+v}\\ +\frac{1}{60}\left(\frac{-11u-11v+10uv+10}{(v-1)(u-1)}\right)g_{n+1}+\frac{1}{360}\left(\frac{u+v+1}{(v-2)(u-2)}\right)g_{n+2}\end{bmatrix} \end{bmatrix}=0 \tag{18}$$

The relevant local truncation error is derived from (18) as follows:

$$L\left[\varphi'\left(t_{n+1}\right);h\right]=-\frac{1}{604800}(-12u-12v+78uv+1) \tag{19}$$

$$L\left[\varphi''\left(t_{n+1}\right);h\right]=-\frac{1}{50400}(-u-v+14uv-1) \tag{20}$$

$$L\left[\varphi''\left(t_{n+1}\right);h\right]=-\frac{1}{50400}(7u+7v+7uv-8) \tag{21}$$

Equating to zero the principal terms of the local truncation errors in (19) – (21) respectively.
Solving (20) and (21) simultaneously, we obtain an optimized value of u and v as

$$u=\frac{16613}{100000},\ v=\frac{85609}{100000} \tag{22}$$

We then Substitute these optimized values of u , $\frac{1}{2}$ and v in (11) to obtain

$$\begin{bmatrix} 12 & 12x_n & 9x_n^2 & 7x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 & \frac{1}{840}x_n^7 & \frac{1}{6720}x_n^8 \\ 0 & 12 & 18x_n & 21x_n^2 & x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 & \frac{1}{840}x_n^7 \\ 0 & 0 & 18 & 42x_n & 3x_n^2 & 3x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 \\ 0 & 0 & 0 & 42 & 6x_n & 3x_n^2 & x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 \\ 0 & 0 & 0 & 42 & 6(x_n+\frac{16613}{100000}h) & 3(x_n+\frac{16613}{100000}h)^2 & (x_n+\frac{16613}{100000}h)^3 & \frac{1}{4}(x_n+\frac{16613}{100000}h)^4 & \frac{1}{20}(x_n+\frac{16613}{100000}h)^5 \\ 0 & 0 & 0 & 42 & 6(x_n+\frac{1}{2}h) & 3(x_n+\frac{1}{2}h)^2 & (x_n+\frac{1}{2}h)^3 & \frac{1}{4}(x_n+\frac{1}{2}h)^4 & \frac{1}{20}(x_n+\frac{1}{2}h)^5 \\ 0 & 0 & 0 & 42 & 6(x_n+\frac{85609}{100000}h) & 3(x_n+\frac{85609}{100000}h)^2 & (x_n+\frac{85609}{100000}h)^3 & \frac{1}{4}(x_n+\frac{85609}{100000}h)^4 & \frac{1}{20}(x_n+\frac{85609}{100000}h)^5 \\ 0 & 0 & 0 & 42 & 6(x_n+h) & 3(x_n+h)^2 & (x_n+h)^3 & \frac{1}{4}(x_n+h)^4 & \frac{1}{20}(x_n+h)^5 \\ 0 & 0 & 0 & 42 & 6(x_n+2h) & 3(x_n+2h)^2 & (x_n+2h)^3 & \frac{1}{4}(x_n+2h)^4 & \frac{1}{20}(x_n+2h)^5 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_0' \\ \mu_0'' \\ \psi_0 \\ \psi_{\frac{16613}{100000}} \\ \psi_{\frac{1}{2}} \\ \psi_{\frac{85609}{100000}} \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} (\varphi_0-\phi_0) \\ (\varphi_0'-\phi_0') \\ (\varphi_0''-\phi_0'') \\ g_n \\ g_{n+\frac{16613}{100000}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{85609}{100000}} \\ g_{n+1} \\ g_{n+2} \end{bmatrix} \tag{23}$$

Using Gaussian elimination method on (23) gives the coefficients of $\mu_0, \mu'_0, \mu''_0, \psi_0, \psi_{\frac{16613}{100000}}, \psi_{\frac{1}{2}}, \psi_{\frac{85609}{100000}}, \psi_1, \psi_2$. The values are subsequently swapped into (3) to get the implicit continuous optimized hybrid Volterra integral equation of the second kinds in the following form;

$$\begin{aligned} p\left((\varphi_n+\xi h)-(\phi_n+\xi h)\right) &= \mu_0(\varphi_n-\phi_n)+h\mu'_0(\varphi'_n-\phi'_n)+h^2\mu''_0(\varphi''_n-\phi''_n) \\ &+h^3\left[\psi_0 g_n+\psi_{\frac{16613}{100000}}g_{n+\frac{16613}{100000}}+\psi_{\frac{1}{2}}g_{n+\frac{1}{2}}+\psi_{\frac{85609}{100000}}g_{n+\frac{85609}{100000}}+\psi_1 g_{n+1}+\psi_2 g_{n+2}\right],\ \xi=\left\{\tfrac{16613}{100000},\tfrac{1}{2},\tfrac{85609}{100000},1,2\right\} \end{aligned} \tag{24}$$

Differentiating (24) first and second time and evaluating at the points $t_{n+\psi}=t_n+\psi h,\ \psi=\left\{0,\frac{16613}{100000},\frac{1}{2},\frac{85609}{100000},1,2\right\}$, then substituting into (1.25) we obtain the discrete optimized hybrid volterra integral equation of the form

$$A^{(0)}B_m^{[1]}=A^{(1)}C_m^{[0]}+\sum_{i=0}D^{[i]}G_m^{[i]}+\sum_{j=u,\frac{1}{2},v,1,2}D^{[j]}G_m^{[j]} \tag{25}$$

Where

$$A^{(0)}=\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix},\ B_m^{[1]}=\begin{bmatrix}
(\varphi_{n+\frac{16613}{100000}}-\phi_{n+\frac{16613}{100000}})\\
(\varphi_{n+\frac{1}{2}}-\phi_{n+\frac{1}{2}})\\
(\varphi_{n+\frac{85609}{100000}}-\phi_{n+\frac{85609}{100000}})\\
(\varphi_{n+1}-\phi_{n+1})\\
(\varphi_{n+2}-\phi_{n+2})\\
(\varphi'_{n+\frac{16613}{100000}}-\phi'_{n+\frac{16613}{100000}})\\
(\varphi'_{n+\frac{1}{2}}-\phi'_{n+\frac{1}{2}})\\
(\varphi'_{n+\frac{85609}{100000}}-\phi'_{n+\frac{85609}{100000}})\\
(\varphi'_{n+1}-\phi'_{n+1})\\
(\varphi'_{n+2}-\phi'_{n+2})\\
(\varphi''_{n+\frac{16613}{100000}}-\phi''_{n+\frac{16613}{100000}})\\
(\varphi''_{n+\frac{1}{2}}-\phi''_{n+\frac{1}{2}})\\
(\varphi''_{n+\frac{85609}{100000}}-\phi''_{n+\frac{85609}{100000}})\\
(\varphi''_{n+1}-\phi''_{n+1})\\
(\varphi''_{n+2}-\phi''_{n+2})
\end{bmatrix}\quad G_m^{(i)}=\begin{bmatrix}
g_{n+\frac{16613}{100000}}\\
g_{n+\frac{1}{2}}\\
g_{n+\frac{85609}{100000}}\\
g_{n+1}\\
g_{n+2}
\end{bmatrix}$$

$$A^{(1)}=\begin{bmatrix} 1 & \frac{16613h}{100000} & \frac{275991769h^2}{20000000000} \\ 1 & \frac{h}{2} & \frac{h^2}{8} \\ 1 & \frac{85609h}{100000} & \frac{7328900881h^2}{20000000000} \\ 1 & h & \frac{h^2}{2} \\ 1 & 2h & 2h^2 \\ 0 & 1 & \frac{16613h}{100000} \\ 0 & 1 & \frac{h}{2} \\ 0 & 1 & \frac{85609h}{100000} \\ 0 & 1 & h \\ 0 & 1 & 2h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
C_m^{[0]}=\begin{bmatrix} (\varphi_n-\phi_n) \\ (\varphi'_n-\phi'_n) \\ (\varphi''_n-\phi''_n) \end{bmatrix}, D^{[i]}=\begin{bmatrix} \frac{14148305502863618094047863052120333h^3}{28764624000000000000000000000000000000} \\ \frac{464840390557h^3}{76458671761920} \\ \frac{161301899639854872141956375002805103h^3}{93032800000000000000000000000000000000} \\ \frac{18785285241h^3}{796444497520} \\ \frac{812702317h^3}{4266666951} \\ \frac{2732019461109837566142585462377h^2}{3595578000000000000000000000000} \\ \frac{115395030227h^2}{4778666985120} \\ \frac{185544764756956693335023396407h^2}{465164000000000000000000000000} \\ \frac{18681598387h^2}{398222248760} \\ \frac{76622537314h^2}{149333343285} \\ \frac{33029453866731456410831039h}{51365400000000000000000000} \\ \frac{10444565729h}{273066684864} \\ \frac{1641292154887966345848673h}{3322600000000000000000000} \\ \frac{137037439h}{2844444634} \\ \frac{5244462317h}{4266666951} \end{bmatrix},$$

$$G_m^{[i]}=[g_n], G_m^{(i)}=\begin{bmatrix} g_{n+\frac{16613}{100000}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{85609}{100000}} \\ g_{n+1} \\ g_{n+2} \end{bmatrix}$$

$$G^{(i)}=\begin{bmatrix}
\frac{20032551100517372502414229312336537h^3}{59180344366498388187840000000000000000} & \frac{-15668983006901384210599501459429227871h^3}{149798588058000000000000000000000000000} & \frac{4356322735166697205289467696930997 5037h^3}{58169486061628780507931808000000000000000} & \frac{-10555250451158487229985962331047925 53h^3}{28800535608000000000000000000000000000} & \frac{35950326662905020846839834603421878 71h^3}{21145644895536000000000000000000000000000} \\
\frac{44280083007812500000000h^3}{3072384565501992884264331} & \frac{465752807h^3}{3423967727040} & \frac{901855468750000000h^3}{1997578504863625 7042559} & \frac{-614124493h^3}{2481276913920} & \frac{961136219h^3}{696660636657152} \\
\frac{623530241344478819913298039306672184 06237h^3}{98316306096063772296458592000000000000000} & \frac{25938615766474659646796624453148148268 83h^3}{106998991470000000000000000000000000000} & \frac{-48119021116865294418320852695 43h^3}{4432345925004728640000000000000000} & \frac{22065122250333424304938343422066046 2727h^3}{33600642876000000000000000000000000000} & \frac{-29155425408682173778914883134046107 97h^3}{14483318421600000000000000000000000000} \\
\frac{28528656250000000000000h^3}{3072384565501992884264331} & \frac{3533283781h^3}{74899294029} & \frac{16634375000000000000000h^3}{6059321464752997969 57623} & \frac{51847927h^3}{168003124380} & \frac{-266643781h^3}{105728224477680} \\
\frac{56827000000000000000000000h^3}{3072384565501992884264331} & \frac{335643000992h^3}{394496470145} & \frac{-20570000000000000000000000h^3}{24901321088026019053053} & \frac{110934224376h^3}{12600234285} & \frac{659972003h^3}{90520740135} \\
\frac{27725218962655755783234172 3493h^2}{36987715229601492617400000000000} & \frac{-38107026147660494852728619562808 99h^2}{187248235072500000000000000000000000} & \frac{58223492843632669794406945208687h^2}{40395476431686653130508200000000000} & \frac{-19711132354353156838839568878121 1h^2}{28000520730000000000000000000000} & \frac{858364028261116106810154684530899h^2}{26432056119420000000000000000000000} \\
\frac{267160683959375000000000h^2}{3072384565501992884264331} & \frac{96020320393h^2}{5991943522320} & \frac{-164042968750000000000h^2}{466101651138485997659 71} & \frac{144152411h^2}{103386538080} & \frac{-336081281h^2}{105728224477680} \\
\frac{1141090270024475321378685719299423 31h^2}{614476913100398576852866200000000000} & \frac{49216556922388110969171841772710589h^2}{374496470145000000000000000000000000} & \frac{139604592074609774092990864 91h^2}{141558047979838520940000000000000} & \frac{-7052182028333750558831309453313 7h^2}{168003124380000000000000000000000} & \frac{-3660014131843392916911796037805 86h^2}{52864112238840000000000000000000000} \\
\frac{686578750000000000000000h^2}{3072384565501992884264331} & \frac{70043500496h^2}{374496470145} & \frac{276212500000000000000000h^2}{6059321464752997969 57623} & \frac{-503652073h^2}{168003124380} & \frac{-44443781h^2}{52864112238840} \\
\frac{-4271000000000000000000000h^2}{653698843723828273247 73} & \frac{961422200992h^2}{374496470145} & \frac{-3730000000000000000000000h^2}{99170564071243829 2893} & \frac{10630263452h^2}{3230829315} & \frac{290489824876h^2}{6608014029855} \\
\frac{1234502093654069080942137 1h}{10567918636874712176400000 0} & \frac{-6252758725722138791318224909 3h}{26749747687500000000000000} & \frac{169035852623891516697408153 91h}{103874082252908536621306800 0000} & \frac{-285244198619041658045005490 93h}{36000669510000000000000000000} & \frac{136834738066048354621371490 93h}{37760080170600000000000000000} \\
\frac{13036238281250000000000h}{4389120807859989834663 33} & \frac{16422037441h}{85599193176} & \frac{-12480898437500000000000h}{259685205632271341553267} & \frac{2422183781h}{115202142432} & \frac{-888873049h}{12083225654592} \\
\frac{45768120779495687551045822 7159h}{17556482321439959338653320 00000} & \frac{104477790934376920612466122 0571h}{26749747867500000000000000000} & \frac{86283943305368099269781h}{44939062850742387600000 0} & \frac{-488395492746174001040572 8127h}{13333581300000000000000000} & \frac{201960175356203325671844594 29h}{37760080170600000000000000000} \\
\frac{11600562500000000000000h}{438912080785998983466333} & \frac{584118014h}{1528557021} & \frac{385625000000000000000h}{137399579699614466 4303} & \frac{6585271h}{266671626} & \frac{5555000h}{188800400853} \\
\frac{-12878200000000000000000000h}{438912080785998983466333} & \frac{62577920000h}{10699899147} & \frac{-26677400000000000000000000h}{259685205632271341553267} & \frac{28444569268h}{3600066951} & \frac{43111226951h}{188800400853}
\end{bmatrix}$$

$$\left(\varphi_{n+\frac{16613}{100000}}-\phi_{n+\frac{16613}{100000}}\right)=\left(\varphi_n-\phi_n\right)+\frac{16613h}{100000}\left(\varphi'_n-\phi'_n\right)+\frac{275991769h^2}{20000000000}\left(\varphi''_n-\phi''_n\right)+\frac{14148305502863618094047863052120333h^3}{2876462400000000000000000000000000000}g_n$$

$$+\frac{20032551100517372502414229312336537h^3}{5918034436649838818784000000000000000}g_{n+\frac{16613}{100000}}-\frac{15668983006901384210599501459429227871h^3}{149798588058000000000000000000000000000}g_{n+\frac{1}{2}}+\frac{43563227351666972052894676969309975037h^3}{5816948606162878050793180800000000000000}g_{n+\frac{85609}{100000}}$$

$$-\frac{1055525045115848722998596233104792553h^3}{288005356080000000000000000000000000000}g_{n+1}+\frac{3595032666290502084683983460342187871h^3}{21145644895536000000000000000000000000000}g_{n+2}$$

$$\left(\varphi_{n+\frac{1}{2}}-\phi_{n+\frac{1}{2}}\right)=\left(\varphi_n-\phi_n\right)+\frac{h}{2}\left(\varphi'_n-\phi'_n\right)+\frac{h}{8}\left(\varphi''_n-\phi''_n\right)+\frac{464840390557h^3}{76458671761920}g_n+\frac{4428008300781250000000h^3}{3072384565501992884264331}g_{n+\frac{16613}{100000}}+\frac{465752807h^3}{3423967727040}g_{n+\frac{1}{2}}$$

$$+\frac{90185546875000000000h^3}{19975785048636257042559}g_{n+\frac{85609}{100000}}-\frac{614124493h^3}{2481276913920}g_{n+1}+\frac{961136219h^3}{696660636657152}g_{n+2}$$

$$\left(\varphi_{n+\frac{85609}{100000}}-\phi_{n+\frac{85609}{100000}}\right)=\left(\varphi_n-\phi_n\right)+\frac{85609h}{100000}\left(\varphi'_n-\phi'_n\right)+\frac{7328900881h^2}{20000000000}\left(\varphi''_n-\phi''_n\right)+\frac{161301899639854872141956375002805103h^3}{930328000000000000000000000000000000000}g_n$$

$$+\frac{62353024134447881991329803930667218406237h^3}{98316306096063772296458592000000000000000}g_{n+\frac{16613}{100000}}+\frac{2593861576647465964679662445314814826883h^3}{1069989914700000000000000000000000000000000}g_{n+\frac{1}{2}}-\frac{4811902111686529441832085269543h^3}{443234592500472864000000000000000000}g_{n+\frac{85609}{100000}}$$

$$+\frac{220651222503334243049383434220660462727h^3}{336006428760000000000000000000000000000}g_{n+1}-\frac{2915542540868217377891488313404610797h^3}{14483318421600000000000000000000000000000}g_{n+2}$$

$$\left(\varphi_{n+1}-\phi_{n+1}\right)=\left(\varphi_n-\phi_n\right)+h\left(\varphi'_n-\phi'_n\right)+\frac{h^2}{2}\left(\varphi''_n-\phi''_n\right)+\frac{18785285241h^3}{796444497520}g_n+\frac{28528656250000000000000h^3}{3072384565501992884264331}g_{n+\frac{16613}{100000}}+\frac{3533283781h^3}{74899294029}g_{n+\frac{1}{2}}$$

$$+\frac{16634375000000000000000h^3}{605932146475299796957623}g_{n+\frac{85609}{100000}}+\frac{51847927h^3}{168003124380}g_{n+1}-\frac{266643781h^3}{105728224477680}g_{n+2}$$

$$\left(\varphi_{n+2}-\phi_{n+2}\right)=\left(\varphi_n-\phi_n\right)+2h\left(\varphi'_n-\phi'_n\right)+2h^2\left(\varphi''_n-\phi''_n\right)+\frac{812702317h^3}{4266666951}g_n+\frac{5682700000000000000000000h^3}{3072384565501992884264331}g_{n+\frac{16613}{100000}}+\frac{335643000992h^3}{394496470145}g_{n+\frac{1}{2}}$$

$$-\frac{2057000000000000000000h^3}{24901321088026019053053}g_{n+\frac{85609}{100000}}+\frac{110934224376h^3}{12600234285}g_{n+1}+\frac{659972003h^3}{90520740135}g_{n+2}$$

$$\left(\varphi'_{n+\frac{16613}{100000}}-\phi'_{n+\frac{16613}{100000}}\right)=\left(\varphi'_n-\phi'_n\right)+\frac{16613h}{100000}\left(\varphi''_n-\phi''_n\right)+\frac{2732019461109837566142585462377h^2}{35955780000000000000000000000000}g_n+\frac{277252189626557557832341723493h^2}{3698771522960149261740000000000}g_{n+\frac{16613}{100000}}$$

$$-\frac{3810702614766049485272861956280899h^2}{187248235072500000000000000000000}g_{n+\frac{1}{2}}+\frac{58223492843632669794406945208687h^2}{40395476431686653130508200000000000}g_{n+\frac{85609}{100000}}-\frac{1971113235435315683888395688781211h^2}{280005207300000000000000000000000}g_{n+1}$$

$$+\frac{858364028261116106810154684530899h^2}{264320561194200000000000000000000}g_{n+2}$$

$$\left(\varphi'_{n+\frac{1}{2}}-\phi'_{n+\frac{1}{2}}\right)=\left(\varphi'_n-\phi'_n\right)+\frac{h}{2}\left(\varphi''_n-\phi''_n\right)+\frac{115395030227h^2}{4778666985120}g_n+\frac{2671606839593750000000000h^2}{3072384565501992884264331}g_{n+\frac{16613}{100000}}+\frac{96020320393h^2}{5991943522320}g_{n+\frac{1}{2}}$$

$$-\frac{16404296875000000000h^2}{46610165113848599765971}g_{n+\frac{85609}{100000}}+\frac{144152411h^2}{103386538080}g_{n+1}-\frac{36081281h^2}{105728224477680}g_{n+2}$$

$$\left(\varphi'_{n+\frac{85609}{100000}}-\phi'_{n+\frac{85609}{100000}}\right)=\left(\varphi'_n-\phi'_n\right)+\frac{85609h}{100000}\left(\varphi''_n-\phi''_n\right)+\frac{185544764756956693335023396407h^2}{4651640000000000000000000000000}g_n+\frac{114109027002447532137868571929942331h^2}{6144769131003985768528662000000000000}g_{n+\frac{16613}{100000}}$$

$$+\frac{49216556922388110969171841772710589h^2}{374496470145000000000000000000000000}g_{n+\frac{1}{2}}$$

$$+\frac{13960459207460977409299086491h^2}{141558047979838520940000000000000}g_{n+\frac{85609}{100000}}-\frac{70521820283337505588313094533137h^2}{168003124380000000000000000000000000}g_{n+1}-\frac{366001413184339291691179603780586h^2}{5286411223884000000000000000000000000}g_{n+2}$$

$$\left(\varphi'_{n+1}-\phi'_{n+1}\right)=\left(\varphi'_n-\phi'_n\right)+h\left(\varphi''_n-\phi''_n\right)+\frac{18681598387h^2}{398222248760}g_n+\frac{6865787500000000000000000h^2}{3072384565501992884264331}g_{n+\frac{16613}{100000}}+\frac{70043500496h^2}{374496470145}g_{n+\frac{1}{2}}$$

$$+\frac{2762125000000000000000h^2}{605932146475299796957623}g_{n+\frac{85609}{100000}}-\frac{503652073h^2}{168003124380}g_{n+1}-\frac{44443781h^2}{52864112238840}g_{n+2}$$

$$\left(\varphi'_{n+2}-\phi'_{n+2}\right)=\left(\varphi'_n-\phi'_n\right)+2h\left(\varphi''_n-\phi''_n\right)+\frac{76622537314h^2}{149333343285}g_n-\frac{42710000000000000000000h^2}{65369884372382827324773}g_{n+\frac{16613}{100000}}+\frac{961422200992h^2}{374496470145}g_{n+\frac{1}{2}}$$

$$-\frac{3730000000000000000000h^2}{991705640712438292893}g_{n+\frac{85609}{100000}}+\frac{10630263452h^2}{3230829315}g_{n+1}+\frac{290489824876h^2}{6608014029855}g_{n+2}$$

$$\left(\varphi''_{n+\frac{16613}{100000}}-\phi''_{n+\frac{16613}{100000}}\right)=\left(\varphi''_n-\phi''_n\right)+\frac{33029453866731456410831039h}{51365400000000000000000000}g_n+\frac{12345020936540690809421371h}{10567918636874712176400000}g_{n+\frac{16613}{100000}}-\frac{62527587257221387913182249093h}{267497476875000000000000000000}g_{n+\frac{1}{2}}+$$

$$\frac{16903585262389151669740815391h}{1038740822529085366213068000000}g_{n+\frac{85609}{100000}}-\frac{28524419861904165804500549093h}{360006695100000000000000000000}g_{n+1}+\frac{13683473806604835462137149093h}{377600801706000000000000000000}g_{n+2}$$

$$\left(\varphi''_{n+\frac{1}{2}}-\phi''_{n+\frac{1}{2}}\right)=\left(\varphi''_{n}-\phi''_{n}\right)+\frac{10444565729h}{273066684864}g_{n}+\frac{16422037441h}{85599193176}g_{n+\frac{16613}{100000}}-\frac{1248089843750000000000h}{2596852056322713415532 67}g_{n+\frac{1}{2}}+\frac{2422183781h}{115202142432}g_{n+\frac{85609}{100000}}-\frac{888873049h}{12083225654592}g_{n+1}-\frac{34\,340625959h}{593406\,599815872}g_{n+2}$$

$$\left(\varphi''_{n+\frac{85609}{100000}}-\phi''_{n+\frac{85609}{100000}}\right)=\left(\varphi''_{n}-\phi''_{n}\right)+\frac{164129215488796634584 8673h}{3322600000000000000000000}g_{n}+\frac{4576812077949568755104582271 59h}{175564823214399593386533200 0000}g_{n+\frac{16613}{100000}}+\frac{10447779093437692061246612205 71h}{26749747867500000000000000000}g_{n+\frac{1}{2}}+\frac{86283943305368099269781h}{4493906285074238760000000}g_{n+\frac{85609}{100000}}-\frac{48839549274617400104057281 27h}{133335813000000000000000000000}g_{n+1}$$
$$+\frac{201960175356203325671844594 29h}{37760080170600000000000000000000}g_{n+2} \tag{26}$$

$$\left(\varphi''_{n+1}-\phi''_{n+1}\right)=\left(\varphi''_{n}-\phi''_{n}\right)+\frac{137037439h}{2844444634}g_{n}+\frac{116005625000000000000000h}{438912080785998983466333}g_{n+\frac{16613}{100000}}+\frac{584118014h}{1528557021}g_{n+\frac{1}{2}}+\frac{38562500000000000000000h}{137399579699614466430 3}g_{n+\frac{85609}{100000}}-\frac{6585271h}{266671626}g_{n+1}+\frac{5555000h}{188800400853}g_{n+2}$$

$$\left(\varphi''_{n+2}-\phi''_{n+2}\right)=\left(\varphi''_{n}-\phi''_{n}\right)+\frac{5244462317h}{4266666951}g_{n}-\frac{128782000000000000000000h}{438912080785998983466333}g_{n+\frac{16613}{100000}}+\frac{62577920000h}{10699899147}g_{n+\frac{1}{2}}-\frac{266774000000000000000000h}{259685205632271341553267}g_{n+\frac{85609}{100000}}+\frac{28444569268h}{3600066951}g_{n+1}+\frac{43111226951h}{188800400853}g_{n+2}$$

2.1 Analysis of the Properties of the Derived Method
Order and Error Constant of the Method

According to Chollom *et.al* [2007], consider the linear difference operator $\mathscr{L}$ be defined using the new technique (26)

$$L\left[\varphi(x);h\right]=\sum_{j=0}^{k}(\varphi-\phi)(x+jh)-h^{3}\left(\sum_{j=0}^{3}\mu_{i}x^{j}+\sum_{j=1}^{5}\psi_{j}e^{x^{j}}\right) \tag{27}$$

Where $(\varphi(x)-\phi(x))$ is the exact solution satisfying equation (27). It can be expressed by Taylor's series expansion around the point x_n to drive the statement

$$\begin{aligned}\ell\left[\left(\varphi(x)-\phi(x)\right):h\right]&=\overline{c_o}\left(\varphi(x)-\phi(x)\right)+\overline{c_1}h\left(\varphi'(x)-\phi'(x)\right)+\overline{c_2}h^{2}\left(\varphi''(x)-\phi''(x)\right)\\&+\overline{c_3}h^{3}\left(\varphi'''(x)-\phi'''(x)\right)+\cdots+\overline{c_{p+3}}h^{p+3}\left(\varphi^{(p+3)}(x)-\phi^{(p+3)}(x)\right)+\cdots\end{aligned} \tag{28}$$

Similarly,
The newly formulated Two-Step Hybrid Optimized Volterra Integral Equation of the second kind (26) is of order p if,

$$\ell\left[\left(\varphi(x)-\phi(x)\right):h\right]=o\left(h^{p+3}\right),\overline{c_o}=\overline{c_1}=\overline{c_2}=\overline{c_3}=\cdots=\overline{c_{p+2}}=0,\overline{c_{p+3}}\neq 0$$

Consequently, the primary local truncation error x_n+k defined as

$$\overline{c_{p+3}}h^{p+3}\left(\varphi-\phi\right)^{p+3}\left(x_n\right)$$

Where

$$C_0=\sum_{j=0}^{k}\mu_j$$

$$C_1=\sum_{j=1}^{k}j\mu_j-\sum_{j=0}^{k}\psi_j$$

$$C_2=\frac{1}{2}\sum_{j=1}^{k}j^{2}\mu_j-\sum_{j=0}^{k}j\psi_j$$

.

. (29)

.

$$C_q=\frac{1}{q!}\sum_{j=1}^{k}j^{q}\mu_j-\frac{1}{(q-1)!}\sum_{j=0}^{k}j^{q-1}\psi_j-\frac{1}{(q-2)!}\sum_{j=0}^{k}j^{q-2}\varpi_j \quad , \quad q=3\ ,\ 4\ ,\ 5\ ,\ .\ .\ .$$

Hence, the new methods (1.26) has order $c_{p+2}=8$, in which its order is $p=\left(6,6,6,6,6,6,6,6,6,6,6,6,6,6\right)^{T}$, with error constant

$$c_9 = \left[-\frac{1313436\,585571426\,493117707\,741574257\,064045183}{453600\,000000000\,000000000\,000000000\,000000000\,000000000}, -\frac{63\,484686679}{2322432000\,000000000}, \frac{2977732\,717769931\,557989089\,665200539\,071569357}{226800\,000000000\,000000000\,000000000\,000000000\,000000000}, \frac{866529637}{36288000\,000000000}, -\frac{31\,466800853}{1134000\,000000000}, \frac{332\,800771432\,895109382\,607591565\,846889181}{6048000000\,000000000\,000000000\,000000000\,000000000}, \frac{465752807}{27648000\,000000000}, \frac{30\,071775335\,194163225\,963618112\,342388151}{288000000\,000000000\,000000000\,000000000\,000000000}, \frac{51847927}{1008000\,000000000}, -\frac{48\,177956219}{378000\,000000000}, -\frac{4605990\,626266000\,708268693\,052389209}{7560\,000000000\,000000000\,000000000\,000000000}, \frac{96\,020320393}{96768000\,000000000}, -\frac{1195140\,952291526\,815032685\,495408019}{2520\,000000000\,000000000\,000000000\,000000000}, -\frac{503652073}{2016000\,000000000}, -\frac{2472\,622456219}{189000\,000000000} \right]$$

Zero Stability

The novel two-step hybrid optimizes the Volterra Integral Equation of the second Kind (26) for k=2, utilizing three off-grid collocation points articulated in the following manner.

$$\rho(z) = \det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i} \right] = 0 \tag{30}$$

Gives

$$\rho(z) = \left| z \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&1&0&0&0&0&\frac{16613}{100000}&0&0&0&0&\frac{275991769}{20\,000000000} \\ 0&0&0&0&1&0&0&0&0&\frac{1}{2}&0&0&0&0&\frac{1}{8} \\ 0&0&0&0&1&0&0&0&0&\frac{85609}{100000}&0&0&0&0&\frac{7328900881}{20\,000000000} \\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&\frac{1}{2} \\ 0&0&0&0&1&0&0&0&0&2&0&0&0&0&2 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&\frac{16613}{100000} \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&\frac{1}{2} \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&\frac{85609}{100000} \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&2 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \right|$$

Finding its determinant and solving the characteristic equation yields

$$= (z-1)^3 z^{12}$$

$$\rho(\mathrm{z}) = (z-1)^3 z^{12} = 0, \quad z = 0,0,0,0,0,0,0,0,0,0,0,0,1,1,1$$

since

$$|z_i| = |0,0,0,0,0,0,0,0,0,0,0,0,1,1,1| \le 1$$

the method is zero-stable.

Consistency of the Method

Definition 1 (see [2024]) "The linear multistep method is said to be consistent if it has order $p \geq 1$. Thus, our method whose order is satisfied this condition

Convergence

For a numerical method to be convergent, it must satisfy both consistency and zero-stability [1962]. Since the three-step optimized hybrid Volterra integral equation of the second kind has been shown to possess these properties, it follows that the method is convergent.

Region of Absolute Stability of the Metod

To delineate the region of absolute stability for the New Two-Step Hybrid Optimized Volterra Integral Equation of the second sort (26), the methodologies were articulated as a generic linear method represented as

$$A^{(0)}\left(\varphi-\phi\right)_m^{(i)}=\sum_{i=0}^{2}e_i\left(\varphi-\phi\right)_n^{(i)}-h^3\left[b_i g\left(\varphi_n-\phi_n\right)+d_i g\left(\varphi_m-\phi_m\right)^w\right] \tag{1.31}$$

$$\left(\varphi-\phi\right)_m^{(i)}=\begin{bmatrix}\left(\varphi-\phi\right)_{n+\frac{16613}{100000}}^{(i)} & \left(\varphi-\phi\right)_{n+\frac{1}{2}}^{(i)} & \left(\varphi-\phi\right)_{n+\frac{85609}{100000}}^{(i)} & \left(\varphi-\phi\right)_{n+1}^{(i)} & \left(\varphi-\phi\right)_{n+2}^{(i)}\end{bmatrix}^T$$

$$\left(\varphi-\phi\right)_n^{(i)}=\begin{bmatrix}\left(\varphi-\phi\right)_{n-\frac{16613}{100000}}^{(i)} & \left(\varphi-\phi\right)_{n-\frac{1}{2}}^{(i)} & \left(\varphi-\phi\right)_{n-\frac{85609}{100000}}^{(i)} & \left(\varphi-\phi\right)_{n-1}^{(i)} & \left(\varphi-\phi\right)_{n}^{(i)}\end{bmatrix}^T$$

$$G\left(\varphi_m-\phi_m\right)=\begin{bmatrix}g_{n+\frac{16613}{100000}}^{(i)} & g_{n+\frac{1}{2}}^{(i)} & g_{n+\frac{85609}{100000}}^{(i)} & g_{n+1}^{(i)} & g_{n+2}^{(i)}\end{bmatrix}^T$$

$$\mathrm{g}\left(\varphi_n-\phi_n\right)=\begin{bmatrix}g_{n-\frac{16613}{100000}}^{(i)} & g_{n-\frac{1}{2}}^{(i)} & g_{n-\frac{85609}{100000}}^{(i)} & g_{n-1}^{(i)} & g_{n}^{(i)}\end{bmatrix}^T$$

Where

and

$A^{(0)}=5\times 5$ Identity matrix

When $i=2$

$$A^{(0)}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\quad e_2=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix},\quad b_0=\begin{bmatrix}0&0&0&0&\frac{33\,029\,453\,866\,731\,456\,410\,831\,039}{513\,654\,000\,000\,000\,000\,000\,000\,000}\\0&0&0&0&\frac{10\,444\,565\,729}{273\,066\,684\,864}\\0&0&0&0&\frac{1641\,292\,154\,887\,966\,345\,848\,673}{33\,226\,000\,000\,000\,000\,000\,000\,000}\\0&0&0&0&\frac{137\,037\,439}{2844\,444\,634}\\0&0&0&0&\frac{5244\,462\,317}{4266\,666\,951}\end{bmatrix}$$

$$d_0=\begin{bmatrix}\frac{12\,345\,020\,936\,540\,690\,809\,421\,371}{105\,679\,186\,368\,747\,121\,764\,000\,000} & -\frac{62\,527\,587\,257\,221\,387\,913\,182\,249\,093}{2674\,974\,786\,750\,000\,000\,000\,000\,000\,000} & \frac{16\,903\,585\,262\,389\,151\,669\,740\,815\,391}{1038\,740\,822\,529\,085\,366\,213\,068\,000\,000} & -\frac{28\,524\,419\,861\,904\,165\,804\,500\,549\,093}{3600\,066\,951\,000\,000\,000\,000\,000\,000\,000} & \frac{13\,683\,473\,806\,604\,835\,462\,137\,149\,093}{377\,600\,801\,706\,000\,000\,000\,000\,000\,000\,000}\\ \frac{130\,362\,382\,812\,500\,000\,000\,000}{438\,912\,080\,785\,998\,983\,466\,333} & \frac{16\,422\,037\,441}{85\,599\,193\,176} & -\frac{12\,480\,898\,437\,500\,000\,000\,000}{259\,685\,205\,632\,271\,341\,553\,267} & \frac{2422\,183\,781}{115\,202\,142\,432} & -\frac{888\,873\,049}{12\,083\,225\,654\,592}\\ \frac{457\,681\,207\,794\,956\,857\,510\,458\,227\,159}{1755\,648\,323\,143\,995\,933\,865\,332\,000\,000} & \frac{1044\,777\,909\,434\,769\,206\,124\,661\,240\,571}{2674\,974\,786\,750\,000\,000\,000\,000\,000\,000} & \frac{86\,283\,943\,305\,368\,099\,269\,781}{449\,390\,628\,507\,423\,876\,000\,000} & -\frac{4883\,954\,927\,461\,740\,010\,405\,728\,127}{133\,335\,813\,000\,000\,000\,000\,000\,000\,000} & \frac{20\,196\,017\,535\,620\,332\,567\,186\,459\,429}{377\,600\,801\,706\,000\,000\,000\,000\,000\,000\,000}\\ \frac{116\,005\,625\,000\,000\,000\,000\,000}{438\,912\,080\,785\,998\,983\,466\,333} & \frac{584\,118\,014}{1528\,557\,021}h & \frac{385\,625\,000\,000\,000\,000\,000}{1373\,995\,796\,996\,144\,664\,303} & \frac{6585\,271}{266\,671\,626} & \frac{5555\,000}{188\,800\,400\,853}\\ -\frac{1287\,820\,000\,000\,000\,000\,000\,000}{438\,912\,080\,785\,998\,983\,466\,333} & \frac{62\,577\,920\,000}{10\,699\,899\,147} & -\frac{2667\,740\,000\,000\,000\,000\,000\,000}{259\,685\,205\,632\,271\,341\,553\,267} & \frac{28\,444\,569\,268}{3600\,066\,951} & \frac{43\,111\,226\,951}{188\,800\,400\,853}\end{bmatrix}$$

Subsequently, calculations were performed utilizing scientific workstation software to derive the stability polynomial for technique (26) utilizing (31).

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}w-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}-h\begin{bmatrix}0&0&0&0&\frac{33\,029\,453\,866\,731\,456\,410\,831\,039}{513\,654\,000\,000\,000\,000\,000\,000\,000}\\0&0&0&0&\frac{10\,444\,565\,729}{273\,066\,684\,864}\\0&0&0&0&\frac{1641\,292\,154\,887\,966\,345\,848\,673}{33\,226\,000\,000\,000\,000\,000\,000\,000}\\0&0&0&0&\frac{137\,037\,439}{2844\,444\,634}\\0&0&0&0&\frac{5244\,462\,317}{4266\,666\,951}\end{bmatrix}$$

$$-w\begin{bmatrix} \frac{12345020936540690809421371}{105679186368747121764000000} & -\frac{62527587257221387913182249093}{2674974786750000000000000000000} & \frac{16903585262389151669740815391}{1038740822529085366213068000000} & -\frac{28524419861904165804500549093}{3600066951000000000000000000000} & \frac{13683473806604835462137149093}{377600801706000000000000000000000} \\ \frac{130362382812500000000000}{438912080785998983466333} & \frac{16422037441}{85599193176} & -\frac{12480898437500000000000}{259685205632271341553267} & \frac{2422183781}{115202142432} & -\frac{888873049}{12083225654592} \\ \frac{457681207794956857510458227159}{1755648323143995933865332000000} & \frac{1044777909434769206124661240571}{2674974786750000000000000000000} & \frac{86283943305368099269781}{449390628507423876000000} & -\frac{4883954927461740010405728127}{133335813000000000000000000000} & \frac{20196017535620332567186459429}{377600801706000000000000000000000} \\ \frac{116005625000000000000000}{438912080785998983466333} & \frac{584118014}{1528557021}h & \frac{385625000000000000000}{1373995796996144664303} & \frac{6585271}{266671626} & \frac{5555000}{188800400853} \\ -\frac{1287820000000000000000000}{438912080785998983466333} & \frac{62577920000}{10699899147} & -\frac{2667740000000000000000000}{259685205632271341553267} & \frac{28444569268}{3600066951} & \frac{43111226951}{188800400853} \end{bmatrix} h$$

Simplifying and finding its determinant gives

$$\begin{aligned} h(w) = {} & h^6\left(\frac{2907609863453814533}{51840964094400000000000}w^5 + \frac{14295777894826071511}{1920035707200000000000}w^4\right) \\ & + h^5\left(\frac{2443700963455275836668656557}{54349704829838152800000000000}w^4 - \frac{286095694404459674137676651}{244573671734271687600000000000}w^5\right) \\ & - h^4\left(\frac{864556516923227721685014157}{2787164350248110400000000000}w^4 - \frac{17438358835318484419677922987}{1760930436486756150720000000000}w^5\right) \\ & - h^3\left(\frac{18657534270045015335711602991}{338640468555145413600000000000}w^5 + \frac{167934449199390303276575410141}{54349704829838152800000000000}w^4\right) \\ & + h^2\left(\frac{136857879436548843003443691269}{550290761402111297100000000000}w^5 + \frac{5158728769929244943705392217}{2264571034576589700000000000}w^4\right) \\ & - h\left(\frac{226111}{300000}w^5 + \frac{373889}{300000}w^4\right) + w^5 - w^4 \end{aligned}$$

The absolute stability region of method (26) is displayed using Mat Lab software.

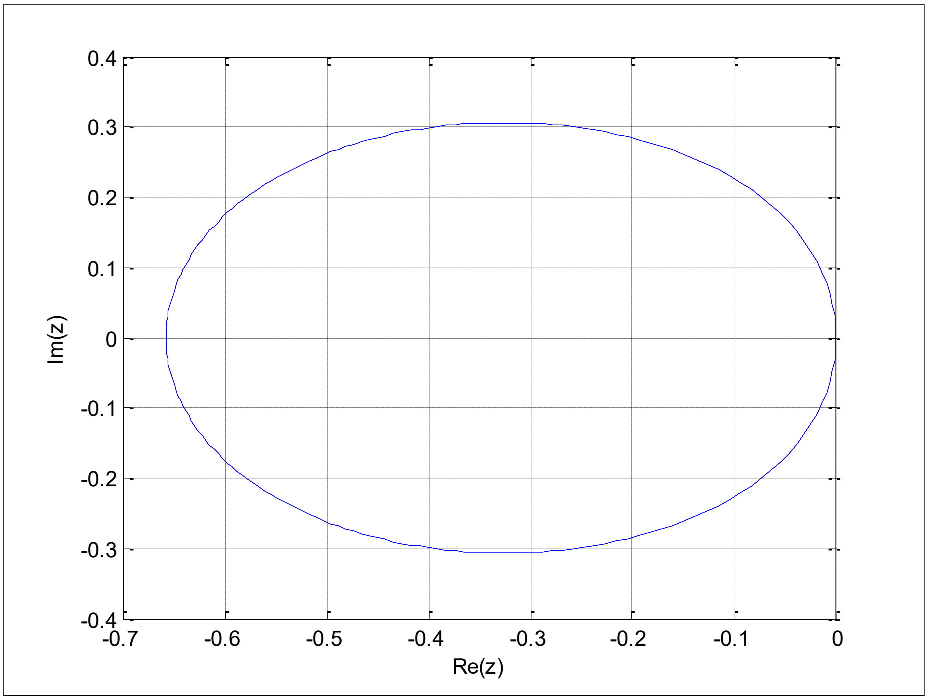

Figure1. Region of Absolute Stability (RAS) for the sixth -order technique (26)

Table 1: Overview of the examination of the methods

Method	Order	Consistency	Zero stability	Error Constant
CASE1 (1.26)	P = 6	Consistent	Zero stable	$C_9 = -\frac{63484686679}{2322432000000000000}$

3. Numerical Results

1 Consider the Stiff problem

$$y''' = -y'$$
$$y(0) = 0\ ,\ y'(0) = 1, y''(0) = 2$$

$$Exact\ Solution;: y(x) = 2(1-\cos x) + \sin x,\ h = \frac{1}{100}$$

Source: Kayode and Adebisi. [2025]

The following notations are used in the tables

X Points of evaluation

Yex Exact Solution

2S3OHM Two step, three optimized hybrid point method

ERR Absolute Error

Table 2: Performance Comparison of exact solution and computed new method for problem 1

X	Yex	2S3OHM
0.1	0.10982508609077662011	0.10982508609077662196
0.2	0.23853617511257795326	0.23853617511257795103
0.3	0.38484722841012753581	0.38484722841012755335
0.4	0.54729635430288032607	0.54729635430288024760
0.5	0.72426041482345756807	0.72426041482345755406
0.6	0.91397124357567876270	0.91397124357567872910
0.7	1.11453331266871420120	1.11453331266871437523
0.8	1.32394267220519191980	1.32394267220519162516
0.9	1.54010697308615447550	1.54010697308615493703
1.0	1.76086637307161707180	1.76086637307161789420

Table 3: Performance Comparison of computed new method and Kayode and Adebisi. [2025] for problem 1

Error in 2S3OHM	Error in A. Victor et al [2022]
1.85000e-18	1.5227819e-12
2.23000e-18	9.6922470e-12
1.75400e-17	2.4267699e-12
7.84700e-17	4.5451198e-11
1.40100e-17	7.8387963e-11
3.36000e-17	1.3159340e-11
1.74000e-16	2.0471091e-10
2.94600e-16	2.9804159e-10
4.61540e-16	4.1925841e-10
8.22400e-16	5.8107297e-10

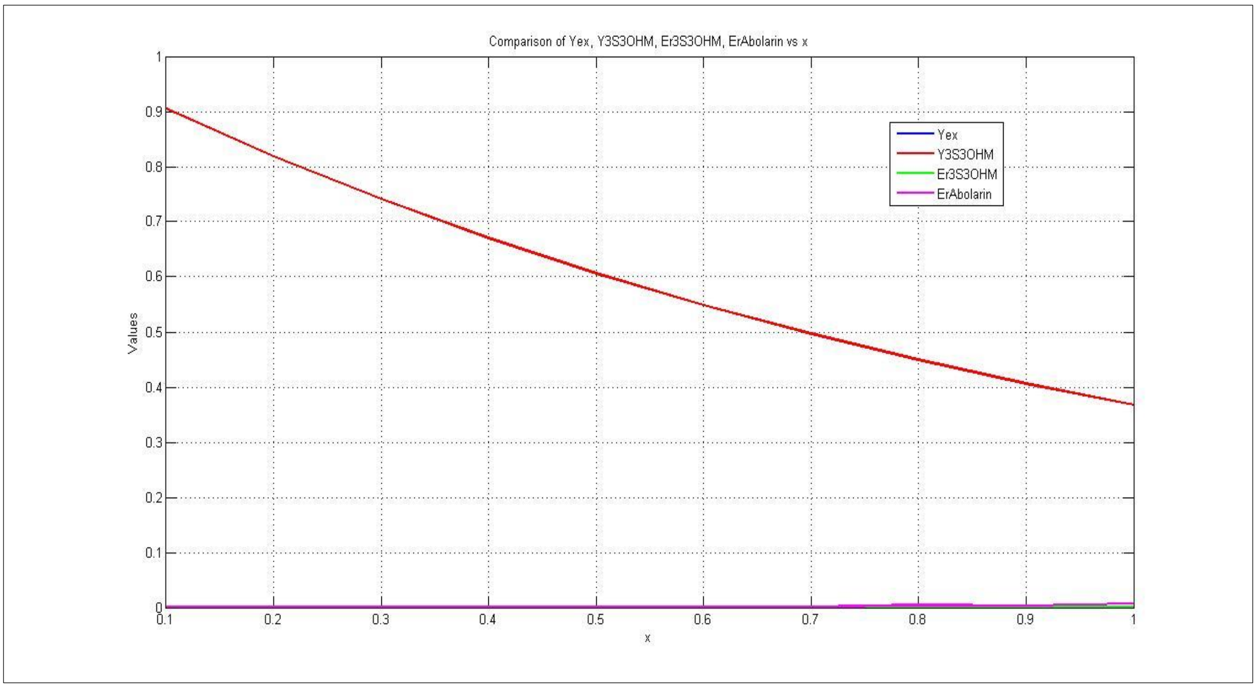

Figure 2: Graphical representation of Performance Comparison of exact solution, Computed new method, the absolute errors in the new method with error in Kayode and Adebisi. [2025]

4. DISCUSSION

Numerical tests were performed to evaluate the efficacy of the Series and Exponentially-Fitted Three-Points Optimized Hybrid Volterra Integral Equation of the Second Kind (SEF-3POH-VIE) approach on typical third-order ordinary differential equations. The findings validate that the approach yields high-order accurate solutions across various step sizes. Expressing the problems as Volterra integral equations of the second kind facilitates direct solution approximation and minimizes cumulative discretization mistakes. Exponential fitting, combined with power series basis functions, is an important factor that increases accuracy in the solution of stiff problems. The proposed scheme has smaller error magnitudes when compared with classical multistep and polynomial based hybrid methods with the same discretization. The hybrid structure with three points of optimality helps in enhanced convergence rates as well as a lower local error. The theoretical order of method is validated numerically and indicates good stability properties. Specifically, the stability region achieved via exponential fitting is enlarged thereby explaining the stability of the performance of the stiff test problems. By such properties, the method is appropriate to practical numerical solution of initial value problems of third order.

5. CONCLUSION

The effectiveness of the Series and Exponentially-Fitted Three-Points Optimized Hybrid Volterra Integral Equation of the Second Kind (SEF-3POH-VIE) to general third-order ordinary differential equations is confirmed by the numerical results in this study. The method gives correct approximations when the step size is considered over all the tested step sizes, and it also gives the best results compared to classical multistep methods and non-fitted hybrid methods in relative terms of the magnitude of the error. By rewriting the governing equations as a Volterra integral equation of the second kind, discrete treatment can be achieved directly and the accumulating discretization error minimized. The combination of power series and exponential fitting basis functions is more effective in the approximation of smooth as well as rapidly varying solutions. The hybrid structure with three points of optimization is a structure that enhances convergence and attains the theoretical order of accuracy as was confirmed by the error decayed under step-size refinement. The numerical experiments also reflect stability properties which are consistent in the theoretical analysis, where exponential fitting expands the stability region and guarantees good performances on stiff and semi-stiff problems. On the whole, the findings are that the given approach is correct, consistent and computationally to perform it.

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