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A Special Case of Central Configuration for Planar Concave 5-body Problem with Newtonian Potentials

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Abstract **Original Research Article**

In this paper we discuss a kind of symmetric configuration for planar concave 5-body problem with four of the five bodies on the vertex of an isosceles trapezoid and the other on the axis of symmetry inside the isosceles trapezoid. Then we give a special case to prove that there exist a set of positive masses to make this configuration a central configuration when the inner body not coincident with the center of mass.

Keywords: Central configurations, n-body problems, celestial mechanics.

INTRODUCTION

From 1687 when Isaac Newton first proposed the 3-body problem in his great work *Philosophiæ Naturalis Principia Mathematica* [1], lots of excellent mathematicians in the history have been attracted to this direction. After the groundbreaking work of Henri Poincaré in the end of the 19th century and the beginning of the 20th century, the study of n-body problem has entered a new era, and many important results have been achieved though it's far from being done. Central configuration is of great significance in the study of this problem, and one of its core problem in this direction is that given any set of positive masses whether or not the classes of central configuration is finite. Naturally we can consider the inverse problem: given a configuration, under what conditions is it possible to choose positive masses that make it a central configuration. More details about N-body problem can be got in [2,3,4,5,6,7,8,9,10,11,12,13].

For N-body problems, According to the Newton's law of motion and the law of gravitation, the i-th particle under the attaction of the other N-1 particles satisfies the 2rd differential equation.

$$m_k\ddot{q}_k(t)=F_k=\sum_{j\neq k,j=1}^{N}\frac{Gm_jm_k(q_j-q_k)}{|q_j-q_k|^3}=\frac{\partial U}{\partial q_k},k=1,\ldots,N \tag{0.1}$$

in which G is the gravitation constant, and

$$U=\sum_{1\le k<j\le N}\frac{Gm_jm_k}{|q_j-q_k|} \tag{0.2}$$

is the Newtonian potential.

The configuration set of this system collected by N particles is $\mathbb{R}^{dN}=\{q=(q_1,q_2,\ldots,q_N)\,|\,q_k\in\mathbb{R}^d,k=1,2,\ldots,N\}$, while the collision set is $\Delta=\{q\in\mathbb{R}^{dN}\,|\,q_j=q_k,j\neq k\}$. Then we call the set $\mathbb{R}^{dN}\setminus\Delta$ a configuration space of the system. To simplify calculation we can choose appropriate dimension to make $G=1$.

Definition 1.1. [14,15,16]If there exists $\lambda\in\mathbb{R}^+$ to make the configuration $q\in\mathbb{R}^{dN}\setminus\Delta$ satisfying

$$-\lambda m_k(q_k-q_c)=\frac{\partial U(q)}{\partial q_k}, \tag{0.3}$$

or

$$-\lambda m_k(q_k - q_c) = \sum_{j\neq k, j=1}^{N} \frac{G m_j m_k (q_j - q_k)}{|q_j - q_k|^3}, \tag{0.4}$$

then we call q a central configuration.

The constant λ is uniquely determined by $\lambda = \dfrac{U(q)}{I(q)}$. The center of mass is, $q_c = \dfrac{\sum_{k=1}^{N} m_k q_k}{\sum_{k=1}^{N} m_k}$, and the moment of inertia is

$I(q) = \sum_{k=1}^{N} m_k |q_k - q_c|^2$.

Specially, if we put the center of mass q_c at the origin, the corresponding configuration space is $X \setminus \Delta$, where $X = \{q = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^{dN} \mid q_c = 0\}$.

In this paper, we study a special case of planar symmetric concave central configuration for 5-body problem with four of the five bodies on the vertex of an isosceles trapezoid and the other on the axis of symmetry inside the isosceles trapezoid.

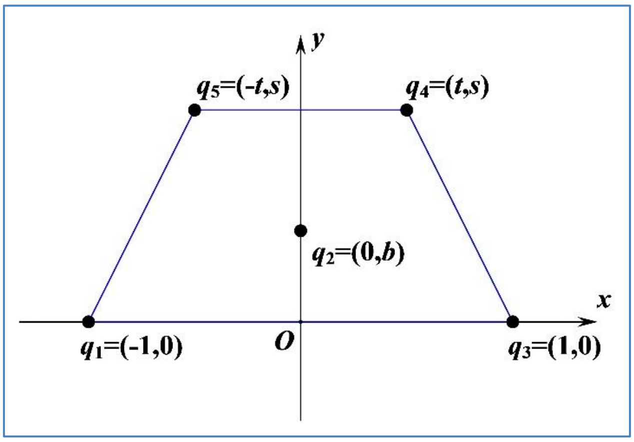

Fig-1: Isosceles trapezoid configuration

Now we establish the cartesian coordinate as shown in **Figure 1**, and the position of the five particles are $q_1 = (-1,0), q_2 = (0,b), q_3 = (1,0), q_4 = (t,s), q_5 = (-t,s)$ Respectively as while as the center of mass $q_c = (c_x, c_y)$. By Definition 1.1, we write
down the equations below.

$$\begin{cases} -\lambda(q_1 - q_c) = m_2 \dfrac{q_2 - q_1}{|q_2 - q_1|^3} + m_3 \dfrac{q_3 - q_1}{|q_3 - q_1|^3} + m_4 \dfrac{q_4 - q_1}{|q_4 - q_1|^3} + m_5 \dfrac{q_5 - q_1}{|q_5 - q_1|^3} \\ -\lambda(q_2 - q_c) = m_1 \dfrac{q_1 - q_2}{|q_1 - q_2|^3} + m_3 \dfrac{q_3 - q_2}{|q_3 - q_2|^3} + m_4 \dfrac{q_4 - q_2}{|q_4 - q_2|^3} + m_5 \dfrac{q_5 - q_2}{|q_5 - q_2|^3} \\ -\lambda(q_3 - q_c) = m_1 \dfrac{q_1 - q_3}{|q_1 - q_3|^3} + m_2 \dfrac{q_2 - q_3}{|q_2 - q_3|^3} + m_4 \dfrac{q_4 - q_3}{|q_4 - q_3|^3} + m_5 \dfrac{q_5 - q_3}{|q_5 - q_3|^3} \\ -\lambda(q_4 - q_c) = m_1 \dfrac{q_1 - q_4}{|q_1 - q_4|^3} + m_2 \dfrac{q_2 - q_4}{|q_2 - q_4|^3} + m_3 \dfrac{q_3 - q_4}{|q_3 - q_4|^3} + m_5 \dfrac{q_5 - q_4}{|q_5 - q_4|^3} \\ -\lambda(q_5 - q_c) = m_1 \dfrac{q_1 - q_5}{|q_1 - q_5|^3} + m_2 \dfrac{q_2 - q_5}{|q_2 - q_5|^3} + m_3 \dfrac{q_3 - q_5}{|q_3 - q_5|^3} + m_4 \dfrac{q_4 - q_5}{|q_4 - q_5|^3} \end{cases} \tag{0.5}$$

Let $r_{ij} = |q_i - q_j|, i \neq j$, and with the symmetry of isosceles trapezoid we have $r_{12} = r_{23}, r_{24} = r_{25}, r_{14} = r_{35}, r_{15} = r_{34}$.
Simplify the equations above we can get

$$\begin{cases} \lambda(c_x+1,c_y)=m_2\dfrac{(1,b)}{r_{12}^3}+m_3\dfrac{(2,0)}{r_{13}^3}+m_4\dfrac{(t+1,s)}{r_{14}^3}+m_5\dfrac{(1-t,s)}{r_{15}^3} \\ \lambda(c_x,c_y-b)=m_1\dfrac{(-1,-b)}{r_{12}^3}+m_3\dfrac{(1,-b)}{r_{12}^3}+m_4\dfrac{(t,s-b)}{r_{24}^3}+m_5\dfrac{(-t,s-b)}{r_{24}^3} \\ \lambda(c_x-1,c_y)=m_1\dfrac{(-2,0)}{r_{13}^3}+m_2\dfrac{(-1,b)}{r_{12}^3}+m_4\dfrac{(t-1,s)}{r_{15}^3}+m_5\dfrac{(-t-1,s)}{r_{14}^3} \\ \lambda(c_x-t,c_y-s)=m_1\dfrac{(-t-1,-s)}{r_{14}^3}+m_2\dfrac{(-t,b-s)}{r_{24}^3}+m_3\dfrac{(1-t,-s)}{r_{15}^3}+m_5\dfrac{(-2t,0)}{r_{45}^3} \\ \lambda(c_x+t,c_y-s)=m_1\dfrac{(t-1,-s)}{r_{15}^3}+m_2\dfrac{(t,b-s)}{r_{24}^3}+m_3\dfrac{(t+1,-s)}{r_{14}^3}+m_4\dfrac{(2t,0)}{r_{45}^3} \end{cases} \tag{0.6}$$

Divide (0.6) equation reference goes here into two parts we get

$$\begin{cases} \lambda(c_x+1)=m_2\dfrac{1}{r_{12}^3}+m_3\dfrac{2}{r_{13}^3}+m_4\dfrac{t+1}{r_{14}^3}+m_5\dfrac{-t+1}{r_{15}^3} \\ \lambda c_x=m_1\dfrac{-1}{r_{12}^3}+m_3\dfrac{1}{r_{12}^3}+m_4\dfrac{t}{r_{24}^3}+m_5\dfrac{-t}{r_{24}^3} \\ \lambda(c_x-1)=m_1\dfrac{-2}{r_{13}^3}+m_2\dfrac{-1}{r_{12}^3}+m_4\dfrac{t-1}{r_{15}^3}+m_5\dfrac{-t-1}{r_{14}^3} \\ \lambda(c_x-t)=m_1\dfrac{-t-1}{r_{14}^3}+m_2\dfrac{-t}{r_{24}^3}+m_3\dfrac{-t+1}{r_{15}^3}+m_5\dfrac{-2t}{r_{45}^3} \\ \lambda(c_x+t)=m_1\dfrac{t-1}{r_{15}^3}+m_2\dfrac{t}{r_{24}^3}+m_3\dfrac{t+1}{r_{14}^3}+m_4\dfrac{2t}{r_{45}^3} \end{cases} \tag{0.7}$$

$$\begin{cases} \lambda c_y=m_2\dfrac{b}{r_{12}^3}+m_4\dfrac{s}{r_{14}^3}+m_5\dfrac{s}{r_{15}^3} \\ \lambda(c_y-b)=m_1\dfrac{-b}{r_{12}^3}+m_3\dfrac{-b}{r_{12}^3}+m_4\dfrac{s-b}{r_{24}^3}+m_5\dfrac{s-b}{r_{24}^3} \\ \lambda c_y=m_2\dfrac{b}{r_{12}^3}+m_4\dfrac{s}{r_{15}^3}+m_5\dfrac{s}{r_{14}^3} \\ \lambda(c_y-s)=m_1\dfrac{-s}{r_{14}^3}+m_2\dfrac{b-s}{r_{24}^3}+m_3\dfrac{-s}{r_{15}^3} \\ \lambda(c_y-s)=m_1\dfrac{-s}{r_{15}^3}+m_2\dfrac{b-s}{r_{24}^3}+m_3\dfrac{-s}{r_{14}^3} \end{cases} \tag{0.8}$$

In (0.8), subtracting the first formula from the third formula and the fourth from the fifth we have $0=(m_4-m_5)\left(\dfrac{s}{r_{14}^3}-\dfrac{s}{r_{15}^3}\right)$ and $0=(m_1-m_3)\left(\dfrac{s}{r_{15}^3}-\dfrac{s}{r_{14}^3}\right)$.

For $r_{14}=\sqrt{(t+1)^2+s^2}\neq\sqrt{(t-1)^2+s^2}=r_{15}, 0<t\leq 1$, we have

$$m_1=m_3, m_4=m_5. \tag{0.9}$$

Substituting (0.9) into the second formula in (0.7) we have

$$c_x=0. \tag{0.10}$$

Then from (0.9) and (0.10) we can simplify (0.7) and (0.8) as below

$$\begin{cases} \lambda = m_1 \dfrac{2}{r_{13}^3} + m_2 \dfrac{1}{r_{12}^3} + m_4 \left(\dfrac{t+1}{r_{14}^3} + \dfrac{-t+1}{r_{15}^3} \right) \\ \lambda t = m_1 \left(\dfrac{t-1}{r_{15}^3} + \dfrac{t+1}{r_{14}^3} \right) + m_2 \dfrac{t}{r_{24}^3} + m_4 \dfrac{2t}{r_{45}^3} \\ \lambda c_y = m_2 \dfrac{b}{r_{12}^3} + m_4 \left(\dfrac{s}{r_{14}^3} + \dfrac{s}{r_{15}^3} \right) \\ \lambda (c_y - b) = m_1 \dfrac{-2b}{r_{12}^3} + m_4 \dfrac{2(s-b)}{r_{24}^3} \\ \lambda (c_y - s) = m_1 \left(\dfrac{-s}{r_{14}^3} + \dfrac{-s}{r_{15}^3} \right) + m_2 \dfrac{b-s}{r_{24}^3} \end{cases} \tag{0.11}$$

RESULTS AND DISCUSSION

Theorem 2.1. If q_2 is not coincident with the center of mass q_c, i.e. $q_2 \neq q_c$, and if $t = 0.5$, then there exists regions G and K (**Figure 1**) in the $b-s$ plane to make the configuration a central configuration.

Firstly by carefully calculating and simplifying we can get the fomulas below from (0.11)

$$m_1 = \lambda \cdot (s-b) \cdot \frac{\left(\dfrac{1}{r_{12}^3} - \dfrac{1}{r_{24}^3} \right) \left(\dfrac{2(s-b)}{r_{24}^3} + \dfrac{b(t+1)-s}{r_{14}^3} + \dfrac{b(-t+1)-s}{r_{15}^3} \right)}{\left(\dfrac{-2(s-b)}{r_{24}^3} - \dfrac{2b}{r_{12}^3} + \dfrac{s}{r_{14}^3} + \dfrac{s}{r_{15}^3} \right) \left(\dfrac{2(s-b)}{r_{13}^3 r_{24}^3} + \dfrac{b(t+1)-s}{r_{12}^3 r_{14}^3} + \dfrac{b(-t+1)-s}{r_{12}^3 r_{15}^3} \right)}, \tag{0.12}$$

$$m_2 = \lambda \cdot \frac{Q}{\left(\dfrac{2(s-b)}{r_{13}^3 r_{24}^3} + \dfrac{b(t+1)-s}{r_{12}^3 r_{14}^3} + \dfrac{b(-t+1)-s}{r_{12}^3 r_{15}^3} \right) \left(\dfrac{-2(s-b)}{r_{24}^3} - \dfrac{2b}{r_{12}^3} + \dfrac{s}{r_{14}^3} + \dfrac{s}{r_{15}^3} \right)}, \tag{0.13}$$

$$m_4 = \lambda \cdot \frac{2b(s-b) \left(\dfrac{1}{r_{24}^3} - \dfrac{1}{r_{12}^3} \right) \left(\dfrac{1}{r_{13}^3} - \dfrac{1}{r_{12}^3} \right)}{\left(\dfrac{-2(s-b)}{r_{24}^3} - \dfrac{2b}{r_{12}^3} + \dfrac{s}{r_{14}^3} + \dfrac{s}{r_{15}^3} \right) \left(\dfrac{2(s-b)}{r_{13}^3 r_{24}^3} + \dfrac{b(t+1)-s}{r_{12}^3 r_{14}^3} + \dfrac{b(-t+1)-s}{r_{12}^3 r_{15}^3} \right)}, \tag{0.14}$$

where

$$\begin{aligned} Q = & \frac{2s(s-b)}{r_{14}^3 r_{24}^3} + \frac{2s(s-b)}{r_{15}^3 r_{24}^3} + \frac{-4s(s-b)}{r_{13}^3 r_{24}^3} + \frac{-2bst}{r_{12}^3 r_{14}^3} + \frac{2bst}{r_{12}^3 r_{15}^3} \\ & + \frac{2s(s-b)}{r_{13}^3 r_{14}^3} + \frac{2s(s-b)}{r_{13}^3 r_{15}^3} + \frac{s(b(t+1)-s)}{r_{14}^6} + \frac{-2s(s-b)}{r_{14}^3 r_{15}^3} + \frac{s(b(-t+1)-s)}{r_{15}^6}, \end{aligned} \tag{0.15}$$

and the equation of t, b, s

$$\begin{aligned} & \frac{2bt(b(t+1)-s)}{r_{12}^3 r_{14}^3} + \frac{2bt(b(-t+1)-s)}{r_{12}^3 r_{15}^3} + \frac{4bt(s-b)}{r_{12}^3 r_{45}^3} + \frac{4bt(s-b)}{r_{13}^3 r_{24}^3} + \frac{(b(t-1)+s)^2}{r_{15}^6} \\ = & \frac{4bt(s-b)}{r_{13}^3 r_{45}^3} + \frac{(b(t+1)-s)^2}{r_{14}^6} + \frac{2(s-b)(b(t+1)-s)}{r_{14}^3 r_{24}^3} + \frac{2(s-b)(b(t-1)+s)}{r_{15}^3 r_{24}^3} \end{aligned} \tag{0.16}$$

With (0.16) we can get some new notifications as below

$$\begin{aligned} a_1 &= \left(\frac{-2(s-b)}{r_{24}^3} - \frac{2b}{r_{12}^3} + \frac{s}{r_{14}^3} + \frac{s}{r_{15}^3} \right) \\ a_2 &= \left(\frac{2(s-b)}{r_{13}^3 r_{24}^3} + \frac{b(t+1)-s}{r_{12}^3 r_{14}^3} - \frac{b(t-1)+s}{r_{12}^3 r_{15}^3} \right), \\ a_3 &= \left(\frac{2(s-b)}{r_{24}^3} + \frac{b(t+1)-s}{r_{14}^3} - \frac{b(t-1)+s}{r_{15}^3} \right) \end{aligned} \tag{0.17}$$

As while as

$$\begin{aligned} Q &= -2sa_2 + sa_3\left(\frac{1}{r_{14}^3}+\frac{1}{r_{15}^3}\right) + 2s(s-b)\left(\frac{1}{r_{12}^3}-\frac{1}{r_{13}^3}\right)\left(\frac{1}{r_{14}^3}+\frac{1}{r_{15}^3}\right) \\ &= \frac{s(s-b)}{bt}\left[\left(\frac{1}{r_{14}^3}-\frac{1}{r_{15}^3}\right)a_3 + 2bt\left(\frac{1}{r_{12}^3}-\frac{1}{r_{13}^3}\right)\left(\frac{1}{r_{14}^3}+\frac{1}{r_{15}^3}+\frac{2}{r_{45}^3}\right)\right] \\ &= \frac{s(s-b)}{bt}Q_1 \end{aligned}$$

Where

$$Q_1 = \left(\frac{1}{r_{14}^3}-\frac{1}{r_{15}^3}\right)a_3 + 2bt\left(\frac{1}{r_{12}^3}-\frac{1}{r_{13}^3}\right)\left(\frac{1}{r_{14}^3}+\frac{1}{r_{15}^3}+\frac{2}{r_{45}^3}\right). \tag{0.18}$$

Then we define

$$\begin{aligned} c_1 &= \sqrt{3}-b \\ c_2 &= t^2+(s-b)^2-b^2-1 \\ c_3 &= s-b \end{aligned} \tag{0.19}$$

And from simple factorization we easily know that $r_{13}^2 - r_{12}^2 = 3-b^2, r_{24}^2 - r_{12}^2 = t^2+(s-b)^2-b^2-1$, so c_1 and $\sqrt{3}-b$, c_2 and $t^2+(s-b)^2-b^2-1$ share the same sign respectively.

Then we get seven new functions $a_1, a_2, a_3, Q_1, c_1, c_2, c_3$ with variables b and s .

Secondly we analysis the necessary and sufficient condition which makes m_1, m_2, m_4 all positive

1) $a_1 a_2 > 0$:

$$(1-1)\begin{cases} Q_1 > 0 \\ a_3 > 0 \\ c_1 > 0 \\ c_2 > 0 \end{cases}, \quad or \quad (1-2)\begin{cases} Q_1 > 0 \\ a_3 < 0 \\ c_1 < 0 \\ c_2 < 0 \end{cases}; \tag{0.20}$$

2) $a_1 a_2 < 0$:

$$(2-1)\begin{cases} Q_1 < 0 \\ a_3 < 0 \\ c_1 < 0 \\ c_2 > 0 \end{cases}, \quad or \quad (2-2)\begin{cases} Q_1 < 0 \\ a_3 > 0 \\ c_1 > 0 \\ c_2 < 0 \end{cases}; \tag{0.21}$$

Thirdly, let $t = 0.5, b \in (0, 2.5), s \in (0, 4)$, and by using computer we can get the regions that make all $m_i > 0 (i = 1, 2, 4)$ positive (see Figure 2).

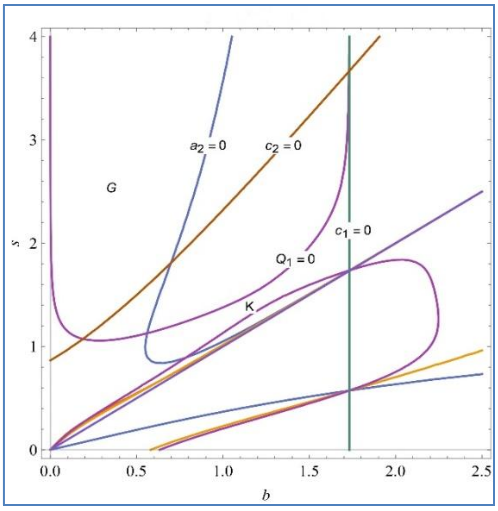

Fig-2: The final regions in plane $b-s$

Finally, we list a table to illustrate the shape of central configuration in region G and region K .

Table-1: Cases of Central configruations in G, K

$t = 0.5$	$m_1 = m_3$	m_2	$m_4 = m_5$	c_y	**Figure**
$(0.3, 1.5) \in G$	1.1559 λ	18.6558 λ	0.8156 λ	0.5816	Fig 3
$(1.6, 2) \in K$	3.7249 λ	0.0554 λ	0.0969 λ	0.1208	Fig 4

Keep the four decimal places.

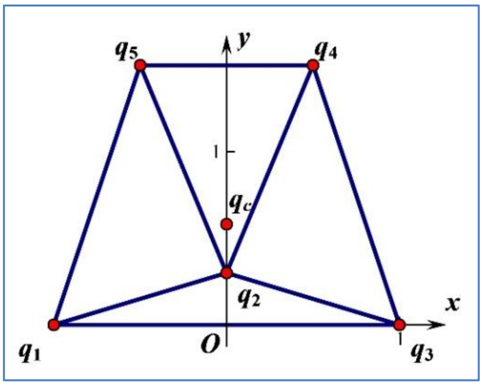

Fig-3: Central configruations in G

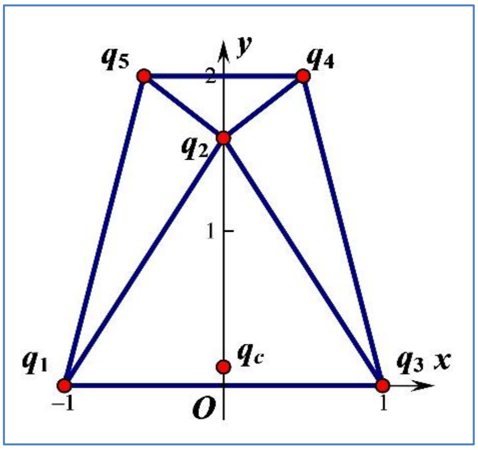

Fig-4: Central configruations in K

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