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An Efficient Approximation for Fractional Differential Equations using Operational Matrix by Hermite Polynomials

Hatice Yalman Kosunalp[1*], Mustafa Gulsu[2]

[1]Bayburt University, Social Sciences Vocational School, Bayburt, Turkey
[2]Mugla Sitki Kocman University, Faculty of Science, Department of Mathematics, Mugla, Turkey

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***Corresponding author:** Hatice Yalman Kosunalp

Abstract **Original Research Article**

This paper aims at the derivation of a new operational matrix of fractional integration for Hermite polynomials, in order to solve the linear form of fractional differential equations (FDEs) in the sense of spectral tau method. To do this, we focus explicitly on the conversion of FDEs into a number of algebraic equations to simplify the problem, subject to pre-defined initial conditions. This is achieved by fractional integration through the Riemann-Liouville sense. We then apply to the proposed strategy to figure out the simplified problem. In order to show the performance of the proposed strategy, we present exact and approximated solutions for a number of examples.

Keywords: Hermite, operational matrix, tau method.

INTRODUCTION

Fractional differential equations (FDEs) have been gainning a significant attention in various aspects of science and engineering [1-5]. FDEs are utilised to describe the mathematical behavior of typical engineering and applied sciences problems. Therefore, a critical effort is required to be placed on the solutions of FDEs. A number of methods have been proposed to solve FDEs [6-10]. It is well-known that many FDEs do not have exact solutions which drives researchers to exploit the numerical techniques for obtaining approximated solutions. A popular technique, called spectral method, uses traditional polynomials to efficiently obtain approximate solutions. Recently, spectral methods have been shown as an attractive subject with a continuously growing interest for a high volume of real-world problems.

Recently, the derivation of operational matrix for diverse types of polynomials was carried out to address the initial value problems of FDEs. Existing studies attempt to solve either linear or non-linear type of FDEs with operational matrix strategy. The operational matrix for fractional integration has been successfully derived for different types of polynomials. Examples of these polynomials are Laguerre [11, 12], Chebyshev [13], Legendre [14], Bessel [15], Bernstein [16], Fermat [17] and Boubaker [18] polynomials. In this study, our main motivation is to establish the operational matrix of fractional integration by Hermite polynomials with the sense of Riemann-Liouville. which is believed to solve multi-order FDEs in a cost-effective way.

The most significant feature of the proposed idea is to reduce the complexity of FDEs. For this purpose, we first write the FDEs in integral type. This is to construct a set of algebraic equations with the operational matrix developed. Depending on the number of initial conditions, a specific number of algebraic equations are also created. By putting all algebraic equations in a matrix form, the problem is actually reduced to the solution of an algrebraic equation system. In order to test the accuracy and performance of the proposed idea, we solved a number of representative examples. With the results obtained, either exact or approximated solutions are achieved. The details about the method and examples are provided in the following sections.

Preliminaries

Definition of Rieman-Liouville Integration

Rieman-Liouville integration that is represented by I^{α}, where $f(x) \in L_1[a,b]$ is introduced as [4]

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \qquad \alpha > 0, x > 0 \,. \tag{2.1}$$

If $\alpha = 0$ then

$$I^0 f(x) = f(x) \tag{2.2}$$

The following properties are hold by Rieman-Liouville operator: [22]

- $I^{\alpha} I^{\beta} = I^{\beta+\alpha}$ (commutative property)
- $I^{\alpha} I^{\beta} = I^{\beta} I^{\alpha}$ (semi-group property)
- $I^{\alpha} x^{c} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+c)} x^{\alpha+c}$ (power function)
- $I^{\alpha}(c_1 f_1 + c_2 f_2) = I^{\alpha}(c_1 f_1) + I^{\alpha}(c_2 f_2)$, $f(x) \in L_1[a,b]$ (linearity)

Method of Operational Matrix

Hermite Polynomials

Hermite polynomials are defined on $(-\infty, \infty)$ with this analytical formula: [20]

$$H_i(x) = \sum_{k=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \frac{(-1)^k i!(2x)^{i-2k}}{k!(i-2k)!} \tag{3.1}$$

where $\left\lfloor \frac{i}{2} \right\rfloor$ denotes the smallest number greatest than $\frac{i}{2}$.

Hermite polynomials are orthogonal since [23]

$$\int_{-\infty}^{\infty} H_i(x) H_j(x) w(x) dx = h_j \delta_{ij} \tag{3.2}$$

where $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ indicates the function of Kronecker and $h_j = 2^j j! \sqrt{\pi}$.

Hermite Operational Matrix Method of Solution

Let Hermite operational matrix is shown as P^{α} with Rieman-Liouville integration and let $\psi(x)$ is the Hermite vector, then it will be

$$I^{\alpha} \psi(x) = P^{(\alpha)} \psi(x) \tag{4.1}$$

Here $P^{(\alpha)}$ is the $(N+1) \times (N+1)$ matrix and matrix elements can be found as below:

$$\varphi(n,k) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{r=0}^{\lfloor k/2 \rfloor} \frac{1}{2^r r! \sqrt{\pi}} \frac{(-1)^{i+r} n! j! 2^{n-2i+k-2r} . \Gamma(n-2i+\alpha+k-2r+1/2)}{2.i! r! (k-2r)! \Gamma(n-2i+v+1)}, \; k = 0,1,2 \ldots N. \tag{4.2}$$

Firstly, for finding the elements of $P^{(\alpha)}$ matrix Rieman-Liouville integration of Hermite polynomials is found, it is

$$I^{\alpha}(H_n(x)) = \sum_{i=0}^{\lceil n/2 \rceil} \frac{(-1)^i \Gamma(n-2i+1) 2^{n-2i+\alpha} [(x)^{n-2i+\alpha}]}{(n-2i)! i! \, \Gamma(n-2i+1+\alpha)} \tag{4.3}$$

Then if we think

$$[(x)^{n-2i+\alpha}] = \sum_{k=0}^{N} c_k H_k(x), \tag{4.4}$$

c_k can be obtained as below

$$c_k = \frac{1}{2^k k! \sqrt{\pi}} \sum_{r=0}^{\lfloor k/2 \rfloor} \frac{(-1)^r k! 2^{k-2r} \Gamma(\frac{n-2i+\alpha+k-2r+1}{2})}{2.r!(k-2r)! \Gamma(n-2i+\alpha+1)} \tag{4.5}$$

By using (4.3) and (4.4) we obtain

$$I^{\alpha}(H_n(x)) = \sum_{k=0}^{N} \varphi(n,k) H_k(x), \quad n = 0,1,2,\ldots,N. \tag{4.6}$$

The (4.6) equation is

$$I^{\alpha}(x)H_n = \sum_{k=0}^{N}\sum_{i=0}^{\lfloor n/2\rfloor}\sum_{r=0}^{\lfloor k/2\rfloor}\frac{1}{2^k k!\sqrt{\pi}}\frac{(-1)^{i+r}n!k!2^{n-2i+k-2r}.\Gamma(n-2i+v+k-2r+1)}{2.i!r!(k-2r)!\Gamma(n-2i+v+1)}H_k(x) \quad (4.7)$$

where

$$\varphi(i,r) = \sum_{i=0}^{\lfloor n/2\rfloor}\sum_{r=0}^{\lfloor k/2\rfloor}\frac{1}{2^j j!\sqrt{\pi}}\frac{(-1)^{i+r}n!k!2^{n-2i+k-2r}.\Gamma(n-2i+v+k-2r+1)}{2.i!r!(k-2r)!\Gamma(n-2i+v+1)}. \quad (4.8)$$

And the fractional differential equation model we will consider is

$$D^{\alpha}u(x) = \sum_{j=1}^{k}\gamma_j D^{\beta_j}u(x) + \gamma_{k+1}u(x) + f(x), \quad (4.9)$$

with initial conditions

$$u^{(i)}(0) = d_i$$

where $\gamma_{j,(j=0,1,2,\dots,k+1)}$ are real constants and $m-1<\alpha\le m$, and $0<\beta_1<\beta_2<\cdots<\beta_k<\alpha$ and $f(x)$ is source function. Rieman-Liouville integral of order α is applied to fractional differential equation

$$u(x) - \sum_{j=0}^{m-1}u^{(j)}(0^+)\frac{x^j}{j!} = \sum_{i=1}^{k}\gamma_i I^{\alpha-\beta_i}\left[u(x) - \sum_{j=0}^{m_i-1}u^{(j)}(0^+)\frac{x^j}{j!}\right] + \gamma_{k+1}I^{\alpha}u(x) + I^{\alpha}f(x) \quad (4.10)$$

$u^{(i)}(0) = d_i, i = 0,\dots,m-1$
where $m_i - 1 < \beta_i \le m_i$, $m_i \in N$. It is obvious that

$$u(x) = \sum_{i=1}^{k}\gamma_i I^{\alpha-\beta_i}u(x) + \gamma_{k+1}I^{\alpha}u(x) + g(x), \quad (4.11)$$

$$u^{(i)}(0) = d_i, i = 0,\dots,m-1$$

where

$$g(x) = I^{\alpha}f(x) + \sum_{j=0}^{m-1}u^{(j)}(0^+)\frac{x^j}{j!} + \sum_{i=1}^{k}\gamma_i I^{\alpha-\beta_i}\left[u(x) - \sum_{j=0}^{m_i-1}u^{(j)}(0^+)\frac{x^j}{j!}\right] \quad (4.12)$$

The matrix forms of $u_N(x)$ and $g(x)$ will be

$$u_N(x) \cong \sum_{i=0}^{N}c_i H_i(x) = C^T\phi(x) \quad (4.13)$$

$$g(x) \cong \sum_{i=0}^{N}g_i H_i(x) = G^T\phi(x) \quad (4.14)$$

where the vector $G = [g_0\ g_1\ \dots\ g_N]^T$ can be calculated whereas $C = [c_0\ c_1\ \dots\ c_N]^T$ is unkown vector.

We then appply Rieman-Liouville integral of order α and $(\alpha - \beta_j)$ of the approximate solution and we obtain

$$I^{\alpha}u_N(x) \cong C^T I^{\alpha}\phi(x) \cong C^T\boldsymbol{P}^{(\alpha)}\varphi(x) \quad (4.15)$$

and

$$I^{\alpha-\beta_j}u_N(x) \cong C^T I^{\alpha-\beta_j}\phi(x) \cong C^T\boldsymbol{P}^{(\alpha-\beta_j)}\varphi(x), \quad j = 1,2,\dots,k. \quad (4.16)$$

The residual $R_N(x)$ will be given as [14,19]

$$R_N(x) = \left(C^T - C^T\sum_{i=1}^{k}\gamma_i I^{\alpha-\beta_i} - \gamma_{k+1}P^{(\alpha)} - G^T\right)\phi(x) \quad (4.17)$$

with Tau method, by applying (Cite ref)

$$\langle R_N(x), H_j(x)\rangle = \int_{-\infty}^{\infty}R_N(x).H_j(x)\,w(x)dx = 0, \quad j = 0,1,\dots,N-m. \quad (4.18)$$

$N-m+1$ linear algebraic equations are generated. From the conditions we have m conditions ,by thinking both of them together we will solve $N+1$ equations with Tau method. [24] The coefficients $[c_0, c_1, \dots, c_N]$ will be found, by putting them into the equation form [25] such as

$$u_N(x) \cong \sum_{i=0}^{N}c_i H_i(x) = C^T\phi(x)$$

Then approximate solutions will be found.

Numerical Examples

Example 1. We first consider the following equation [26]

$$D^{0.5}u(x) + u(x) = x^2 + \frac{2x^{1.5}}{\Gamma(2.5)}\ u(0) = 0, 0 < x < 1. \quad (5.1)$$

where the exact solution is given as

$$u(x) = x^2$$

Our solution is applied for N=2 and the approximate solution is written as

$$u_N(x) = \sum_{i=0}^{2} c_i H_i(x) = C^T\,\phi(x) \text{ and } g(x) = \sum_{i=0}^{2} g_i H_i(x) = G^T\,\phi(x)$$

We also obtain

$$P^{0.5} = \begin{bmatrix} 0.3901 & 0.2885 & 0.0488 \\ 0.3847 & 0.3901 & 0.1443 \\ -0.1580 & 0.1923 & 0.2925 \end{bmatrix}$$

The following algebraic equations are extracted, one from the initial condition:

$$\begin{aligned} 1.3901c_0 + 0.3847c_1 - 0.1560c_2 &= 0.6560 \\ 0.2885c_0 + 1.3901c_1 + 0.1923c_2 &= 0.1923 \\ c_0 - 2c_2 &= 0 \end{aligned}$$

The solution of this algebraic equations is

$$[c_0 \quad c_1 \quad c_2] = [0.5 \quad 0 \quad 0.25]$$

Finally, our proposed method achieves the exact solution by:

$$u_N(x) = [c_0 \quad c_1 \quad c_2] \begin{bmatrix} H_0(x) \\ H_1(x) \\ H_2(x) \end{bmatrix} = x^2$$

Example 2. The second example is **[12]**

$$D^{5/2}u(x) + 3u(x) = 3x^3 + \frac{6x^{1/2}}{\Gamma(1.5)}, \quad u(0) = 0, u'(0) = 0, u''(0) = 0 \qquad (5.2)$$

The exact solution is given as

$$u(x) = x^3$$

We also obtain the following

$$P^{2.5} = \begin{bmatrix} 0.0780 & 0.0962 & 0.0488 & 0.0120 \\ 0.0550 & 0.0780 & 0.0481 & 0.0163 \\ -0.0867 & -0.0824 & -0.0195 & 0.0080 \\ -0.2098 & -0.2600 & -0.1237 & -0.0195 \end{bmatrix}$$

The required algebraic equations are found, three of which come from the initial conditions:

$$\begin{aligned} 1.2340c_0 + 0.1649c_1 - 0.2600c_2 - 0.6295c_3 &= 0.0449 \\ 8c_2 &= 0 \\ c_0 - 2c_2 &= 0 \\ 2c_1 - 12c_3 &= 0 \end{aligned}$$

Now after applying our technique for $N = 3$ we get

$$[c_0 \quad c_1 \quad c_2 \quad c_3] = [0 \quad 0.75 \quad 0 \quad 0.125]$$

And the approximate solution is the same as exact solution like

$$u_N = u(x) = x^3$$

Example 3. The third example is [27]

$$D^2u(x) + D^{0.75}u(x) + u(x) = x^3 + 6x + \frac{8.53333x^{2.25}}{\Gamma(0.25)}, \quad u(0) = 0, u'(0) = 0 \qquad (5.3)$$

The exact solution is given as:

$$u(x) = x^3$$

We obtain the following fractional integrations for N = 3,

$$P^2 = \begin{bmatrix} 0.1250 & 0.1410 & 0.0625 & 0.0118 \\ 0.0940 & 0.1250 & 0.0705 & 0.0208 \\ -0.1250 & -0.0940 & 0 & 0.0235 \\ -0.3385 & -0.3750 & -0{,}1410 & 0 \end{bmatrix}$$

$$P^{1.25} = \begin{bmatrix} 0.2345 & 0.2232 & 0.0733 & 0.0047 \\ 0.1984 & 0.2345 & 0.1116 & 0.0244 \\ -0.1804 & -0.0496 & 0.0879 & 0.0651 \\ -0.6303 & -0.5411 & -0{,}0744 & 0.0879 \end{bmatrix}$$

The required algebraic equations are composed, two of which come from initial conditions.

$$1.3595c_0 + 0.2925c_1 - 0.3054c_2 - 0.9688c_3 = 0.1196$$
$$1.3643c_0 + 1.3595c_1 - 0.1436c_2 - 0.9161c_3 = 0.8754$$
$$c_0 - 2c_2 = 0$$
$$2c_1 - 12c_3 = 0$$

Solving these algrebraic equations, the coefficients are obtained as

$$[c_0 \quad c_1 \quad c_2 \quad c_3] = [0.0209 \quad 0.7204 \quad 0.0104 \quad 0.1201]$$

Then, the approximated solutiın for this example is found as

$$u_N(x) = [c_0 \quad c_1 \quad c_2 \quad c_3] \begin{bmatrix} H_0(x) \\ H_1(x) \\ H_2(x) \\ H_3(x) \end{bmatrix} = 0.96x^3 + 0.041x^2$$

We see from the approximated solution that it converges to the exact solution very closely, which is presented in figure 1 below.

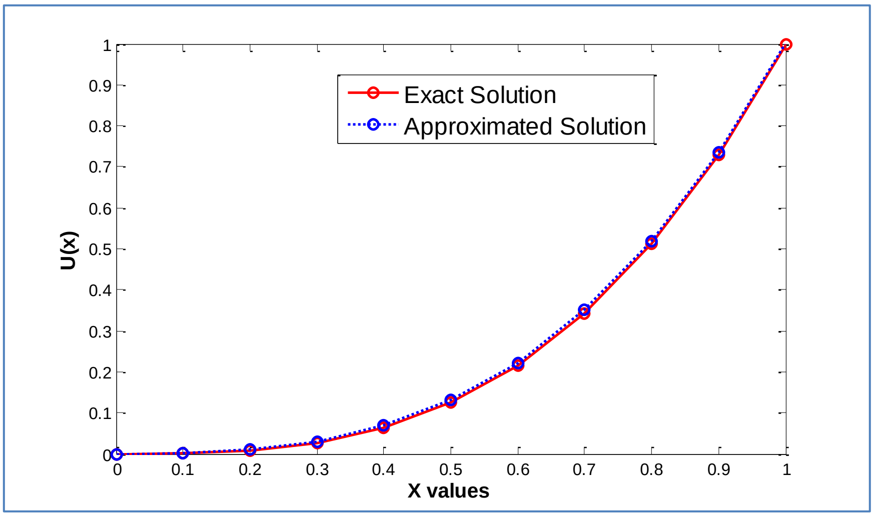

Fig-1: The results of the exact and approximated solutions.

CONCLUSİONS

This paper presented an explicit derivation of operational matrix of fractional integration through Hermite polynomials. It relies basically on solutions of fractional differential equations with Riemann-Liouville. We considered the linear type of fractional differential equations with initial conditions. The basic mechanism is based on the conversion of FDEs into a set of algebraic equations for simplicity. We proved the performance of the proposed mechanism with a number of numerical examples, presenting exact and approximated solutions with high accuracy.

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