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Bounded Traveling Wave Solutions of the (3+1)-Dimensional Calogero-Bogoyavlenskii-Schiff Equation

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Abstract **Original Research Article**

In this paper, the (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation is studied by the bifurcation theory of dynamical system. Based on this theory, phase portraits of different topological structures of the equation are obtained, which clearly show all bounded orbits corresponding to the bounded traveling waves of the equation. Furthermore, the periodic wave solution of the **(**3+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation are obtained by calculating complicated elliptic integrals.

Keywords: Traveling wave, elliptic integral, dynamical system.

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1. INTRODUCTION

In the past four decades, the research area of nonlinear evolution equations modeling various physical phenomena has played a significant role in a great many applications such as fluid mechanics and water waves. A large amount of effort has been expended over the last ten years or so in attempting to find robust and stable analytical methods to solve these equations. Many powerful methods have been presented to investigate exact solutions of nonlinear equations, such as the Backlünd transformation method [1, 2], the homogeneous balance method [3], Jacobi elliptic function method [4], extended tanh method [5, 6], F-expansion method [7, 8], Lie group analysis [9-11], the modified simple equation method [12, 13], variational iteration method [14], and so no.

In 1990, Bogoyavlenskii and Schiff used the nonlinear integrable equation Calogero-Bogoyavlenskii-Schiff (CBS) equation to describe the interaction of Riemann waves along a two-dimensional space [15, 16]. Riemann wave mechanics is one of the most important applications of physics and engineering, such as tsunamis and tides in rivers, magneto acoustic waves in plasmas, internal waves in oceans, and optical tsunamis in fibers.

In this paper, we study the following (3+1)-dimensional CBS equation

$$u_{xt}+u_xu_{xy}+u_{xx}u_y+u_xu_{xz}+u_{xx}u_z+u_{xxxy}+u_{xxxz}=0, \quad \ldots\ldots\ldots\ldots\ldots (1.1)$$

At present, scholars have published a lot of research results on the solution of Calogero-Bogoyavlenskii-Schiff (CBS) equation. For example, multiple Exp-function method is used to obtain multiple soliton solutions of CBS equation [16], and multiple soliton solutions and cross solutions are constructed based on Bell polynomial, auxiliary variables and bilinear form [17]. There are also many research methods, such as the singular popular method, the generalized Kudryashov method, the modified simple equation method, the symmetric method and the generalized Riccarty equation expansion method [19-23].

Although there are many profound consequences about the traveling wave solutions of Eq. (1.1), which are beneficial for us to understanding of nonlinear physical phenomena and wave propagation, the traveling wave solutions of Eq. (1.1) is not sufficient discussed, especially for its bounded traveling wave solutions. Therefore, the purpose of this paper is to find all possible bounded traveling wave solutions in Eq. (1.1). Motivated by them, our first step is to transform the traveling wave equation of Eq. (1.1) into a dynamical system in R^3. Fortunately, we can find a 2-dimensional invariant manifold which determines most of dynamical behavior. Then, bifurcation analysis can be applied to seek the parameter bifurcation sets which determine various qualitatively different phase portraits. Finally, by calculating the complex elliptic integrals along these orbits, we will show the expressions of all bounded traveling wave solutions in the (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation.

2. Traveling wave system and bifurcation analysis

With the following traveling wave transformation
$u = u(t, x, y, z) = u(\xi) = u(x + ay + bz - ct)$,

Equation (1.1) can be transformed into its raveling wave system

$$-cu'' + 2(a + b)u'u'' + (a + b)u'''' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.1)}$$

Where $'$ stands for $d/d\xi$, $a, b \neq 0$ represent the wave numbers in the y and z directions respectively and $c \neq 0$ is the wave speed. Integrating (2.1) once and retaining an integral constant, we have

$$(a + b)u''' + (a + b)(u')^2 - cu' = e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.2)}$$

Where parameter e is the integral constant, letting $u' = v$, we have

$$\begin{cases} u' = v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . \text{(2.3)} \\ v'' = -v^2 + \dfrac{c}{a + b}v + \dfrac{e}{a + b} \ \ldots\ldots \text{(2.4)} \end{cases}$$

Obviously, Eq. (2.4) does not contain function u. So let us analyze the flow of Eq. (2.4) firstly. Without a doubt, Eq. (2.4) can be rewrite to the equivalent system

$$\begin{cases} v' = y \\ y' = -v^2 + \frac{c}{a+b}v + \frac{e}{a+b} \end{cases} \ldots\ldots\ldots\ldots \textit{(2.5)}$$

Which is exactly a Hamiltonian system with the energy function

$$H(v, y) = \frac{1}{2}y^2 + \frac{1}{3}v^3 - \frac{cA}{2}v^2 - eAv \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.6)}$$

Where $A = \frac{1}{a+b}$.
Next, we need to discuss the equilibrium of system (2.4).

Theorem 2.1. When $c^2A^2 + 4eA > 0$, system (2.5) has two equilibria, a saddle $E_1\left(\frac{Ac-\sqrt{c^2A^2+4eA}}{2}, 0\right)$ and a center $E_2\left(\frac{Ac+\sqrt{c^2A^2+4eA}}{2}, 0\right)$. When$c^2A^2 + 4eA = 0$, system (2.5) has a unique equilibrium of higher order $E_3\left(\frac{Ac}{2}, 0\right)$, which is a cusp. When $c^2A^2 + 4eA < 0$, system (2.5) has no equilibrium.

Proof. When $c^2A^2 + 4eA > 0$, a direct calculation shows that system (2.5) has two equilibria $E_1\left(\frac{Ac-\sqrt{c^2A^2+4eA}}{2}, 0\right)$ and $E_2\left(\frac{Ac+\sqrt{c^2A^2+4eA}}{2}, 0\right)$. Let $M_i(i = 1, 2, 3)$ to denote the Jacobi matrix of system (2.5) at point $E_i(i = 1, 2, 3)$, we have

$$M_1 = \begin{bmatrix} 0 & 1 \\ \sqrt{c^2A^2 + 4eA} & 0 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 0 & 1 \\ -\sqrt{c^2A^2 + 4eA} & 0 \end{bmatrix}.$$

From this, it is not different for us to check

$$det\, M_1 = -\sqrt{c^2A^2 + 4eA} < 0,$$
$$det\, M_2 = \sqrt{c^2A^2 + 4eA} > 0.$$

By the theory of plane dynamic system [24, 25, 26] and the properties of Hamiltonian system [25], it is not difficult to check that E_1 is a saddle and E_2 is a center.

When $c^2A + 4e = 0$, the system (2.5) has only one equilibrium $E_3\left(\frac{Ac}{2}, 0\right)$,with a nilpotent matrix

$$M_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This shows that E_3 is a degenerated equilibrium. In order to judge the type of E_3 further, we do the following homeomorphic transformation

$$\alpha = v - \frac{Ac}{2}, \beta = y,$$

At this point, the system (2.5) can be transformed into its normal form below

$$\begin{cases} \alpha' = \beta, \\ \beta' = -\alpha^2 + \dfrac{c^2A^2}{4} + eA. \end{cases}$$

By the qualitative theory of differential equation [26], we have $k = 2$ and $b_n = 0$, which means that E_3 is a cusp.

When $c^2A^2 + 4eA < 0$, it is easy to see that there is no equilibrium of system (2.5).

Obviously, the hypersurface $\{(a, b, c, e)|c^2A^2 + 4eA = 0\}$ divides the 4-dimensional parameter space into two regions. The corresponding parameter bifurcation sets are composed of $\{(a, b, c, e)|c^2A^2 + 4eA > 0\}$, $\{(a, b, c, e)|c^2A^2 + 4eA = 0\}$ and$\{(a, b, c, e)|c^2A^2 + 4eA < 0\}$.To illustrate the parameter bifurcation sets ,we fix the parameters at a=1 and b=0 to give a special bifurcation boundary.

$$L: e = \frac{c^2}{4}$$

Shown in Fig 1

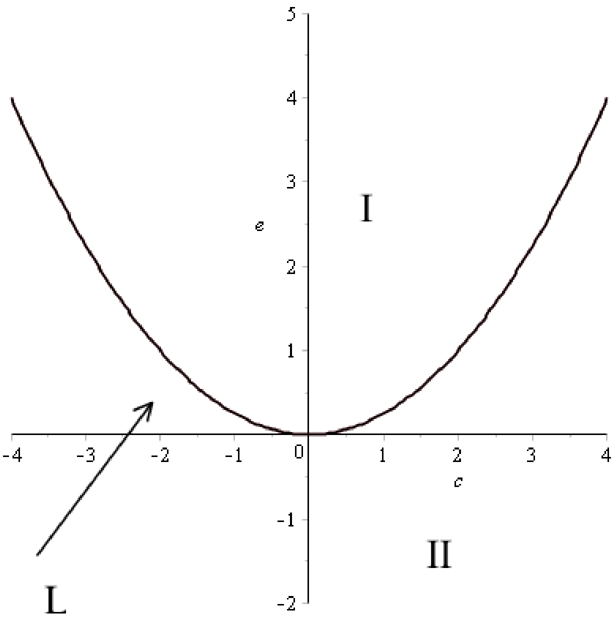


Fig 1: Transition boundary on c-e plane

As we know, the Hamiltonian system is a system determined by its potential energy function. So, according the energy function (2.6) and the properties of the Hamiltonian system [19], we have the following results.

Case 1: Consider $c^2A^2 + 4eA > 0$, there is a homoclinic orbit γconnected to the saddle E_1. The center E_2 is surrounded by the family of periodic orbits

$$\Gamma(h) = \{H(v, y) = h, h \in (h(E_2,), h(E_1,))\},$$

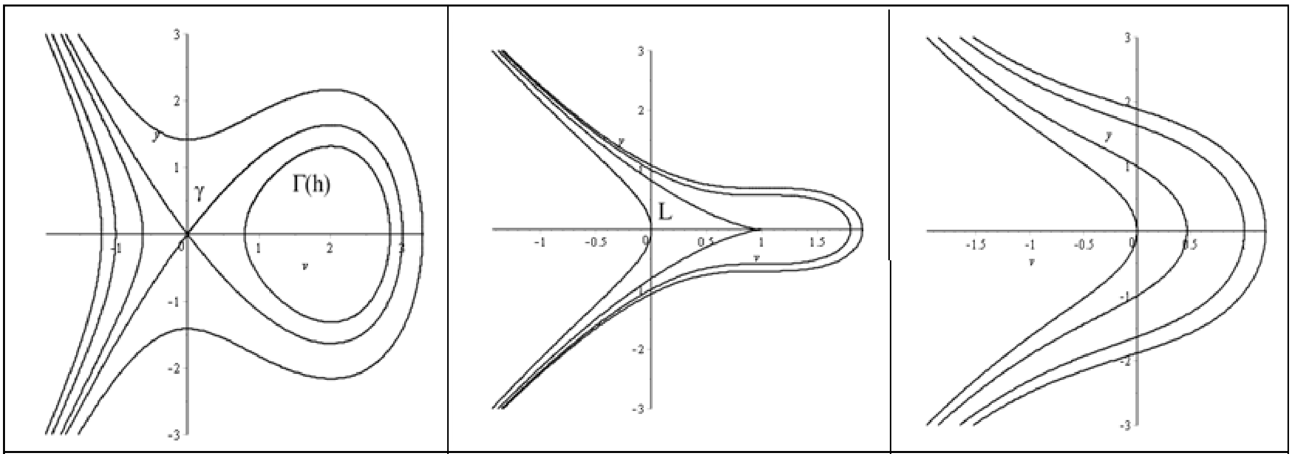
Where

$$h(E_1) = \frac{-A^3c^3 + (c^2A^2 + 4eA)\sqrt{c^2A^2 + 4eA} - 6ecA^2}{12},$$
$$h(E_2) = \frac{-A^3c^3 - (c^2A^2 + 4eA)\sqrt{c^2A^2 + 4eA} - 6ecA^2}{12}.$$

Moreover, $\Gamma(h)$ tends to E_2 as $h \to h(E_2)$ and tends to γ as $h \to h(E_1)$, besides the homoclinic orbit and periodic orbits, other orbits of system (2.5) are unbounded, as shown in Fig 2(a).

Case 2: When $c^2A^2 + 4eA = 0$, all the orbits here were unbounded, the system (2.5) has two types of orbits. Orbit L was different from other orbits, as show in Fig 2(b).

Case 3: When $c^2A^2 + 4eA < 0$, all the orbits here were unbounded, the system (2.5) has only one type of orbits, as show in Fig 2(c).



(a) (b) (c)

Fig 2: The phase portraits of (2.4)

Obviously, there is only case 1 has bounded orbits, namely a family of periodic orbits $\Gamma(h)$ and a homologous orbit γ (see fig.2(a)), which correspond to the periodic wave and shock wave of system (2.5) respectively. Then we will give the expressions of traveling wave solutions corresponding to these bounded orbitals by calculating complicated elliptic integrals.

3. Explicit traveling wave solutions of Eq. (1.1)

In this section, we will give the explicit expression of all bounded traveling wave solutions for Eq. (1.1). According to the system (2.5), in order to derive the final traveling wave solutions $u(\xi)$ of the (3+1)-dimensional CBS equation, we need to integrate the solutions of system (2.5) once with respect to ξ.

3.1 Consider the periodic orbits, from the energy function (2.5), any one of the periodic orbits $\Gamma(h)$ can be expressed by

$$y = \pm\sqrt{\frac{2}{3}}\sqrt{(v_3 - v)(v - v_1)(v - v_2)},$$

Where v_1, v_2 and v_3 are real numbers and the relations $v_1 < v_2 < v < v_3$ hold. Assume that the period of these closed orbits is 2T, and choose initial value $v(0) = v_2$, we have

$$\int_{v_2}^{v} \frac{dv}{\sqrt{\frac{2}{3}}\sqrt{(v_3 - v)(v - v_1)(v - v_2)}} = \int_0^{\xi} d\xi\,, 0 < \xi < T.$$

$$\int_{v}^{v_2} \frac{dv}{-\sqrt{\frac{2}{3}}\sqrt{(v_3 - v)(v - v_1)(v - v_2)}} = \int_{\xi}^{0} d\xi\,, -T < \xi < 0.$$

The two integral expressions can be rewritten as

$$\int_{v_2}^{v} \frac{dv}{\sqrt{\frac{2}{3}}\sqrt{(v_3 - v)(v - v_1)(v - v_2)}} = |\xi|, -T < \xi < T.$$

Noting that

$$\int_{v_2}^{v} \frac{dv}{\sqrt{(v_3 - v)(v - v_1)(v - v_2)}} = g \cdot sn^{-1}\left(\sqrt{\frac{(v_3 - v_1)(v - v_2)}{(v_3 - v_1)(v - v_1)}}, k\right),$$

Where $k^2 = \frac{v_3 - v_2}{v_3 - v_1}$ and $g = \frac{2}{\sqrt{v_3 - v_1}}$, we get the expression of periodic wave solution of the system (2.5)

$$v_1(\xi) = v_1 + \frac{(v_3 - v_1)(v_2 - v_1)}{(v_3 - v_1) - (v_3 - v_2)sn^2\left(\sqrt{\frac{v_3 - v_1}{6}}|\xi|\right)}, -T < \xi < T.$$

The odevity of elliptic function leads to

$$v_1(\xi) = v_1 + \frac{(v_2 - v_1)}{1 - \frac{v_3 - v_2}{v_3 - v_1} sn^2\left(\sqrt{\frac{v_3 - v_1}{6}}\xi\right)}, -T < \xi < T.$$

From (2.3), we need to integral above expression once again to get the final solution of Eq. (1.1) using the integral formula of elliptic function

$$\int \frac{du}{1 \pm k \cdot sn(u)} = \frac{1}{k'^2}\{E(u) + k[1 \mp k \cdot sn(u)]cd(u)\}$$

Where $k' = \sqrt{1 - k^2}$.

Then, the first type of bounded traveling wave solution of system (1.1) can be calculated as follows

$$u_1(\xi) = \int v_1(\xi)\, d\xi = \int \left[v_1 + \frac{(v_2 - v_1)}{1 - \frac{v_3 - v_2}{v_3 - v_1} sn^2\left(\sqrt{\frac{v_3 - v_1}{6}}\xi\right)}\right] d\xi$$

$$= v_1\xi + \frac{v_2 - v_1}{2}\int \left[\frac{1}{1 - k \cdot sn\left(\sqrt{\frac{v_3 - v_1}{6}}\xi\right)} + \frac{1}{1 + k \cdot sn\left(\sqrt{\frac{v_3 - v_1}{6}}\xi\right)}\right] d\xi$$

$$= v_1\xi + \sqrt{6(v_3 - v_1)}\left[E\left(\sqrt{\frac{v_3 - v_1}{6}}\xi\right) + k \cdot cd(\sqrt{\frac{v_3 - v_1}{6}}\xi)\right]$$

Where $k^2 = \frac{v_3 - v_2}{v_3 - v_1}$ and $-T < \xi < T$.

3.2. Consider the homologous orbit whose energy is equal to the energy of E_1. In fact, it is a homologous orbit of system (2.5) and can be expressed by

$$v = \pm\sqrt{\frac{2}{3}}(v - v_4)\sqrt{v_5 - v},$$

Where the relation $-\infty < v_4 < v < v_5$ holds, and $v_4 = \frac{Ac - \sqrt{c^2A^2 + 4eA}}{2}$, $v_5 = \frac{Ac + 2\sqrt{c^2A^2 + 4eA}}{2}$, letting initial value $v(0) = v_5$, we have

$$\int_{v}^{v_5} \frac{dv}{\sqrt{\frac{2}{3}}(v - v_4)\sqrt{v_5 - v}} = \int_{\xi}^{0} d\xi, \xi < 0,$$

$$\int_{v_5}^{v} \frac{dp}{-\sqrt{\frac{2}{3}}(v - v_4)\sqrt{v_5 - v}} = \int_{0}^{\xi} d\xi, \xi > 0,$$

Which can be rewritten as

$$\int_{v_5}^{v} \frac{dp}{\sqrt{\frac{2}{3}}(v - v_4)\sqrt{v_5 - v}} = -|\xi|, -\infty < \xi < +\infty.$$

Noting that

$$\int_{v_5}^{v} \frac{dv}{(v-v_4)\sqrt{v_5-v}} = -\frac{1}{\sqrt{v_5-v_4}} \ln \frac{\sqrt{v_5-v_4}+\sqrt{v_5-v}}{\sqrt{v_5-v_4}-\sqrt{v_5-v}},$$

We get the expression of solitary wave solution of the system (2.5)

$$v_2(\xi) = v_4 + \frac{4(v_5-v_4)\cdot \exp(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}|\xi|)}{\left(1+\exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}|\xi|\right)\right)^2}, -\infty < \xi < +\infty.$$

Note that when $\xi < 0$, we have

$$\begin{aligned} v_2(\xi) &= v_4 + \frac{4(v_5-v_4)\cdot \exp\left(-\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)}{\left(1+\exp\left(-\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)\right)^2} \\ &= v_4 + \frac{4(v_5-v_4)\cdot \exp\left(-\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)\cdot \exp\left(2\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)}{\left(1+\exp\left(-\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)\right)^2\cdot \exp\left(2\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)} \\ &= v_4 + \frac{4(v_5-v_4)\cdot \exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)}{\left(1+\exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)\right)^2} \end{aligned}$$

It means that $v_2(\xi)$ has the same form for whether $\xi > 0$ or $\xi < 0$, It means that $v_2(\xi)$ can be simplified to the following form

$$v_2(\xi) = v_4 + \frac{4(v_5-v_4)\cdot \exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)}{\left(1+\exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)\right)^2}, -\infty < \xi < +\infty.$$

Then, the second type of bounded traveling wave solution of Eq. (1.1) can be calculated by

$$\begin{aligned} u_2(\xi) = \int v_2(\xi)\, d\xi &= \int \left(v_4 + \frac{4(v_5-v_4)\cdot \exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)}{\left(1+\exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)\right)^2} \right) d\xi \\ &= v_4\xi - \frac{2\sqrt{6}\sqrt{v_5-v_4}}{1+\exp\left(\sqrt{\frac{2}{3}}\sqrt{v_5-v_4}\xi\right)} + C_1, \end{aligned}$$

Where $-\infty < \xi < +\infty$ and C_1 is a constant.

4. CONCLUSIONS

In this paper, we apply the dynamical system methods to investigate all bounded traveling waves of the (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation. Although it is a high dimensional dynamical system, we find that there exists a 2-dimensional Hamiltonian system which determines the most of the dynamical behavior. And then we completely investigate all bounded orbits of it by detailed analyzing the phase space geometry, and all possible bounded traveling waves of the (3+1)-dimensional CBS equation and corresponding existence conditions can be identified clearly. Last, using complex elliptic function, we get the traveling solutions.

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