New Concrete Constructions of Nullnorms on Bounded Lattices
Zheng Xu

1 College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610025, Sichuan, P. R. China

*Corresponding author: Zheng Xu
College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610025, Sichuan, P. R. China

Abstract

The structure of the nullnorms are the basis for the study of nullnorms. This paper presents two concrete methods to construct nullnorms via triangular subconorms (triangular subnorms) and triangular norms (triangular conorms) on bounded lattices, then gets two constructions of nullnorms on bounded lattices via triangular subconorms (triangular subnorms) and triangular norms (triangular conorms).

Keywords: nullnorms; bounded lattices; triangular subconorms; triangular norms.

1. INTRODUCTION

The concept of nullnorm on unit interval [0,1] was introduced by Calvo [1]. From a theoretical point of view, nullnorm is important. Meanwhile, it is also widely used in many fields, such as expert systems, fuzzy quantifiers, neural networks, fuzzy logic [2].

Since bounded lattices [3] are more general than unit intervals [2-9], most studies of nullnorms focus on bounded lattices [10-12]. Based on the existence of t-norms and t-conorms on bounded lattices, Karaçal et al. [10] defined nullnorms on bounded lattices and proposed three construction methods of nullnorms on bounded lattices with an arbitrary zero element a ∈ L \ {0,1}. Later, some construction methods of nullnorms on bounded lattices were also proposed by Ertuğr et al., [11, 19, 20]. For the first time, Xie, Ji [18] constructed nullnorms via triangular subconorms (triangular subnorms) on bounded lattices.

In order to complete the structure of nullnorms on bounded lattices, two concrete methods to construct nullnorms via triangular subconorms (triangular subnorms) and triangular norms (triangular conorms) on bounded lattices are presented in this paper.

2. Preliminaries

In this section, we will recall some basic definitions and theorems which will be applied to this paper.

Definition 2.1. [13] A lattice \( (L, \leq) \) is bounded if it has top and bottom elements, which are written as 1 and 0, respectively; that is, two elements 0,1 ∈ L exist such that 0 ≤ x ≤ 1 for all x ∈ L.

Throughout this paper, unless stated otherwise, we denote L as a bounded lattice with the top and bottom elements 1 and 0, respectively.

Definition 2.2. [13] Given a bounded lattice \( (L, \leq, 0,1) \) and \( a, b \in L \), \( a \leq b \), a subset \( [a,b] \) of L is defined as \( [a,b] = \{ x \in L | a \leq x \leq b \} \). Similarly, denote \( (a,b) = \{ x \in L | a < x < b \} \) and \( (a,b) = \{ x \in L | a < x < b \} \). If a and b are incomparable, we use the notation \( a \nparallel b \). The set of all elements which are incomparable with a are denoted by \( I_a \).

Definition 2.3. [14] Let \( (L, \leq, 0,1) \) be a bounded lattice.

(1) An operation \( T : L^2 \rightarrow L \) is called a triangular norm (t-norm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element 1 ∈ L such that \( T(x,1) = x \) for all \( x \in L \).
(2) An operation \( S: L^2 \rightarrow L \) is called a triangular conorm (t-conorm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element \( 0 \in L \) such that \( S(x,0) = x \) for all \( x \in L \).

**Definition 2.4.**[15] Let \((L, \leq, 0, 1)\) be a bounded lattice. A commutative, associative, non-decreasing in each variable function \( V: L^2 \rightarrow L \) is called a nullnorm if an element \( a \in L \) exists such that \( V(x,0) = x \) for all \( x \leq a \) and \( V(x,1) = x \) for all \( x \geq a \).

It is easy to see that \( V(x,a) = a \) for all \( x \in L \), thus \( a \) is the zero element for \( V \).

**Theorem 2.1.**[16] Let \((L, \leq, 0, 1)\) be a bounded lattice and \( V: L^2 \rightarrow L \) be a nullnorm on \( L \) with the zero element \( a \). Then,

\[
V_T^S = \begin{cases} 
S(x,y) & (x,y) \in [0,a]^2 \\
T(x,y) & (x,y) \in [a,1]^2 \\
S(x \land a, y \land a) & (x,y) \in [0,a] \times I_a \cup I_a \times [0,a] \cup I_a \times I_a \\
a & \text{otherwise},
\end{cases}
\]

\[
V_S^T = \begin{cases} 
S(x,y) & (x,y) \in [0,a]^2 \\
T(x,y) & (x,y) \in [a,1]^2 \\
T(x \land a, y \land a) & (x,y) \in [a,1] \times I_a \cup I_a \times [a,1] \cup I_a \times I_a \\
a & \text{otherwise},
\end{cases}
\]

And they are nullnorms on \( L \) with zero element \( a \).

In order to reduce the complexity in the proof of associativity, we introduce the following theorem.

**Theorem 2.3.**[21] Let \( S \) be a nonempty set and \( A, B, C, D \) be subsets of \( S \). Let \( H \) be a commutative binary operation on \( S \). Then \( H \) is associative on \( A \cup B \cup C \cup D \) both of the following statements hold:

1. \( H(H(x,y),z) = H(H(x,y,z)) \) for all \((x,y,z) \in (A,A,A) \cup (B,B,B) \cup (C,C,C) \cup (D,D,D) \cup (A,A,B) \cup (A,B,B) \cup (A,A,C) \cup (A,C,C) \cup (A,A,D) \cup (A,D,D) \cup (B,B,C) \cup (B,C,C) \cup (B,B,D) \cup (B,D,D) \cup (C,C,D) \cup (C,D,D) \).

2. \( H(H(x,y),z) = H(H(x,z),y) \) for all \((x,y,z) \in (A,A,B) \cup (A,B,D) \cup (A,C,D) \cup (B,C,D) \).

3. New Constructions of Nullnorms on Bounded Lattices

In this section, we will recall some basic definitions and theorems which will be applied to this paper.

© 2022 Scholars Journal of Physics, Mathematics and Statistics | Published by SAS Publishers, India
Theorem 3.1. Let \((L, \leq, 0, 1)\) be a bounded lattice. \(a \in L \setminus \{0, 1\}\), \(R\) is a \(t\)-subconorm on \([0, a]\), and \(T\) is a \(t\)-norm on \([a, 1]\). Then, the function \(V_T^R : L^2 \rightarrow L\) can be defined as:

\[
V_T^R = \begin{cases} 
R(x, y) & (x, y) \in (0, a]^2 \\
x \land y & (x, y) \in \{0\} \times [0, a] \cup [0, a] \times \{0\} \\
T(x, y) & (x, y) \in [a, 1]^2 \\
(x \land a) \lor (y \land a) & (x, y) \in [0, a] \times I_a \cup I_a \times [0, a] \cup I_a \times I_a \\
a & \text{otherwise,}
\end{cases}
\]

And it is nullnorm on \(L\) with zero element \(a\), if and only if \(x \land a = 0\) for all \(x \in I_a\).

Proof. Sufficiency: The commutativity of \(V_T^R\) can be proven directly based on its description. Similarly, we can express \(V_T^R(x, 0) = x\) for all \(x \in [0, a]\) and \(V_T^R(x, 1) = x\) for all \(x \in [a, 1]\). Now, we only need to proof monotonicity and associativity.

Monotonicity: Let us prove that if \(x \leq y\), then \(V_T^R(x, z) \leq V_T^R(y, z)\) for all \(z \in L\).

1. It is obvious that \(V_T^R(x, z) \leq V_T^R(y, z)\), if \(x = 0\).

2. \(x \in (0, a]\)
   2.1. \(y = (0, a]\)
       2.1.1. \(z = 0\)
           \(V_T^R(x, z) = x \leq y = V_T^R(y, z)\)
       2.1.2. \(z \in (0, a]\)
           \(V_T^R(x, z) = R(x, z) \leq R(y, z) = V_T^R(y, z)\)
       2.1.3. \(z \in [a, 1]\)
           \(V_T^R(x, z) = a = V_T^R(y, z)\)
       2.1.4. \(z \in I_a\)
           \(V_T^R(x, z) = (x \land a) \lor (z \land a) = x \leq y = (y \land a) \lor (z \land a) = V_T^R(y, z)\)

2.2. \(y \in [a, 1]\)
   2.2.1. \(z = 0\)
       \(V_T^R(x, z) = x \leq a = V_T^R(y, z)\)
   2.2.2. \(z \in (0, a]\)
       \(V_T^R(x, z) = R(x, z) \leq a = V_T^R(y, z)\)
   2.2.3. \(z \in [a, 1]\)
       \(V_T^R(x, z) = a \leq T(y, z) = V_T^R(y, z)\)
   2.2.4. \(z \in I_a\)
       \(V_T^R(x, z) = (x \land a) \lor (z \land a) = x \leq a = V_T^R(y, z)\)

3. \(x \in [a, 1]\)
   3.1. \(y \in [a, 1]\)
3.1.1. \( z = 0 \)
\[ V^R_T(x, z) = a = V^R_T(y, z) \]
3.1.2. \( z \in (0, a] \)
\[ V^R_T(x, z) = a = V^R_T(y, z) \]
3.1.3. \( z \in [a, 1] \)
\[ V^R_T(x, z) = T(x, z) \leq T(y, z) = V^R_T(y, z) \]
3.1.4. \( z \in I_a \)
\[ V^R_T(x, z) = a = V^R_T(y, z) \]
4. \( x \in I_a \)

4.1. \( y \in I_a \)

4.1.1. \( z = 0 \)
\[ V^R_T(x, z) = (x \land a) \lor (z \land a) = 0 = (y \land a) \lor (z \land a) = V^R_T(y, z) \]
4.1.2. \( z \in (0, a] \)
\[ V^R_T(x, z) = (x \land a) \lor (z \land a) = z = (y \land a) \lor (z \land a) = V^R_T(y, z) \]
4.1.3. \( z \in [a, 1] \)
\[ V^R_T(x, z) = a = V^R_T(y, z) \]
4.1.4. \( z \in I_a \)
\[ V^R_T(x, z) = (x \land a) \lor (z \land a) = 0 = (y \land a) \lor (z \land a) = V^R_T(y, z) \]

4.2. \( y \in [a, 1] \)

4.2.1. \( z = 0 \)
\[ V^R_T(x, z) = (x \land a) \lor (z \land a) = 0 \leq a = V^R_T(y, z) \]
4.2.2. \( z \in (0, a] \)
\[ V^R_T(x, z) = (x \land a) \lor (z \land a) = z \leq a = V^R_T(y, z) \]
4.2.3. \( z \in [a, 1] \)
\[ V^R_T(x, z) = a \leq T(y, z) = V^R_T(y, z) \]
4.2.4. \( z \in I_a \)
\[ V^R_T(x, z) = (x \land a) \lor (z \land a) = 0 \leq a = V^R_T(y, z) \]

Associativity: It can be shown that \( V^R_T\left(V^R_T(x, y), z\right) = V^R_T\left(x, V^R_T(y, z)\right) \) for all \( x, y, z \in L \). By Theorem 2.3, we only need to consider the following cases:

1. \( x = 0, y = 0, z = 0 \)
\[ V^R_T\left(V^R_T(x, y), z\right) = V^R_T(0, z) = 0 = V^R_T(x, 0) = V^R_T\left(x, V^R_T(y, z)\right) \]
2. \( x \in (0, a], y \in (0, a], z \in (0, a] \)
\[ V^R_T\left(V^R_T(x, y), z\right) = V^R_T(R(x, y), z) = R(R(x, y), z) \]
\[ = R(x, R(y, z)) = V^R_T(x, R(y, z)) = V^R_T\left(x, V^R_T(y, z)\right) \]
3. \( x \in [a, 1], y \in [a, 1], z \in [a, 1] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(T(x, y), z) = T(T(x, y), z)
= T(x, T(y, z)) = V^R_T(x, T(y, z)) = V^R_T \left( x, V^R_T(y, z) \right)
\]

4. \( x \in I_a, y \in I_a, z \in I_a \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(0, z) = 0 = V^R_T(x, 0) = V^R_T \left( x, V^R_T(y, z) \right)
\]

5. \( x = 0, y = 0, z \in (0, a] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(0, z) = z = V^R_T(x, z) = V^R_T \left( x, V^R_T(y, z) \right)
\]

6. \( x = 0, y \in (0, a], z \in (0, a] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(y, z) = R(y, z) = V^R_T(x, R(y, z)) = V^R_T \left( x, V^R_T(y, z) \right)
\]

7. \( x = 0, y = 0, z \in [a, 1] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(0, z) = a = V^R_T(x, a) = V^R_T \left( x, V^R_T(y, z) \right)
\]

8. \( x = 0, y \in [a, 1], z \in [a, 1] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(a, z) = a = V^R_T(x, T(y, z)) = V^R_T \left( x, V^R_T(y, z) \right)
\]

9. \( x = 0, y = 0, z \in I_a \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(0, z) = 0 = V^R_T(x, 0) = V^R_T \left( x, V^R_T(y, z) \right)
\]

10. \( x = 0, y \in I_a, z \in I_a \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(x, z) = 0 = V^R_T(x, 0) = V^R_T \left( x, V^R_T(y, z) \right)
\]

11. \( x \in (0, a], y \in (0, a], z \in [a, 1] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(R(x, y), z) = a = V^R_T(x, a) = V^R_T \left( x, V^R_T(y, z) \right)
\]

12. \( x \in (0, a], y \in [a, 1], z \in [a, 1] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(a, z) = a = V^R_T(x, T(y, z)) = V^R_T \left( x, V^R_T(y, z) \right)
\]

13. \( x \in (0, a], y \in (0, a], z \in I_a \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(R(x, y), z) = R(x, y) = V^R_T(x, y) = V^R_T \left( x, V^R_T(y, z) \right)
\]

14. \( x \in (0, a], y \in I_a, z \in I_a \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(x, z) = x = V^R_T(x, 0) = V^R_T \left( x, V^R_T(y, z) \right)
\]

15. \( x \in [a, 1], y \in [a, 1], z \in I_a \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(T(x, y), z) = a = V^R_T(x, a) = V^R_T \left( x, V^R_T(y, z) \right)
\]

16. \( x \in [a, 1], y \in I_a, z \in I_a \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(a, z) = a = V^R_T(x, 0) = V^R_T \left( x, V^R_T(y, z) \right)
\]

17. \( x = 0, y \in (0, a], z \in [a, 1] \)
\[
V^R_T \left( V^R_T(x, y), z \right) = V^R_T(y, z) = a = V^R_T(x, a) = V^R_T \left( x, V^R_T(y, z) \right)
\]

18. \( x = 0, y \in (0, a], z \in I_a \)
In previous studies, nullnorms on bounded lattices have been defined and studied extensively. Moreover, the concrete construction of nullnorm on bounded lattices is still an active research field.