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KKM Type Theorems in GFC-Spaces and Their Applications

Rui-Jiang Bi[1*], Rong-Hua He[2]

[1,2]College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, Sichuan, P. R. China

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***Corresponding author:** Rui-Jiang Bi
College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, Sichuan, P. R. China

Abstract **Review Article**

In this paper, GFC-KKM maps with compactly closed (or compactly open) values are established in GFC-space, and some KKM-type theorems are given. Finally, as applications, some fixed point theorems and inequalities are proved. These theorems generalize some related results in the recent literature.

Keywords: GFC-space; GFC-KKM mapping; KKM-type theorems.

INTRODUCTION

In 1929, Knaster and other three Polish mathematicians initiated the research on KKM theory, and later many people improved and developed the KKM theorem in many aspects. Since 1987, after Horvath [11] gave H-space without linear structure by replacing convex hull with contractible set, Park and Kim [16] introduced G-convex spaces, Ben-El-Mechaiekh *et al.*, [13] introduced L-convex spaces, Ding [14] introduced FC-space without any convexity structure, Khanh [2] introduced GFC-space, and in this paper, GFC-KKM maps are established in GFC-space, and some KKM-type theorems are given. Finally, as applications, some fixed point theorems and inequalities are proved. These theorems generalize some related results in the recent literature.

1. Preliminaries

Let X be a nonempty set. We denote by 2^X and $\langle X\rangle$ the family of all subsets of X and the family of all nonempty finite subsets of X. Let Δ_n be the standard (n-1) simplex with vertices $\{e_1,e_2,\dots,e_n\}$ in R^n. For any nonempty subset J of $\{1,2,\dots,n\}$, let $\Delta_J=(\{e_j: j\in J\})$.

Definition 1.1. (*[2]*)

Let X be a topologic space, Y be a nonempty set, and Φ be a family of continuous mappings $\varphi:\Delta_n\to X$, $n\in N$. Then a triple (X,Y,Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) if for each finite subset $N=\{y_1,y_2,\dots,y_n\}\in\langle Y\rangle$, there is $\varphi_N:\Delta_n\to X$ of the family Φ.

Definition 1.2. (*[8]*)

Let (X,Y,Φ) be a GFC-space and $T:Y\to 2^X$ is a multivalued mapping. T is said to be a GFC-KKM mapping if for any $N=\{y_1,y_2,\dots,y_n\}\in\langle Y\rangle$ there exists $\varphi_N\in\Phi$ such that for any $\{y_{i_1},y_{i_2},\dots,y_{i_k}\}\in\{y_1,y_2,\dots,y_n\}$, we have

$$\varphi_N(\Delta_k)\subset\bigcup_{j=1}^{k}T(y_{i_j}).$$

Definition 1.3. (*[8]*)

Let (X,Y,Φ) be a GFC-space, $S,T:Y\to 2^X$ are two set-valued mappings. S is called relative GFC-convex with respect to T, if $\forall y\in Y$, $\forall N=\{y_1,y_2,\cdots y_n\}\in\langle Y\rangle$, $\forall\{y_{i_1},y_{i_2},\cdots y_{i_k}\}\in\langle N\cap T(y)\rangle$, then $\varphi_N(\Delta_k)\subset S(y)$.

Definition 1.4. (*[7]*)

Let (X,Y,Φ) be a GFC-space, Y is a nonempty subset of X, and D is a nonempty subset of Y. D is called to GFC-convex, if $\forall N=\{y_1,y_2,\cdots y_n\}\in\langle Y\rangle$, $\forall\{y_{i_1},y_{i_2},\cdots y_{i_k}\}\in\langle N\cap D\rangle$, then we have $\varphi_N(\Delta_k)\subset D$.

Let A be a subset of X. A is said to be compactly open (or compactly closed) in X if for each nonempty compact subset K of X, $A\cap K$ is open (or closed) in K.

Let X and Y be two sets and $S:Y\to 2^X$ be a set-valued mapping, then $S^{-1}:X\to 2^Y$ and $S^*:X\to 2^Y$ are defined as $S^{-1}(x)=\{y\in Y:x\in S(y)\}$ and $S^*(x)=Y\setminus S^{-1}(x)$, respectively. Obviously, $y\in S^*(x)$ when and only when $x\notin S(y)$.

2. GFC-KKM Type Theorems

Theorem 2.1. Let (X,Y,Φ) be a GFC-space, and $T:Y\to 2^X$ be a set-valued mapping with compactly closed values.

1. If T is a GFC-KKM mapping, then the family $\{T(y):y\in Y\}$ has the finite intersection property.
2. If the family $\{T(y):y\in Y\}$ has the finite intersection property, there exists $x\in X$ such that $x\in\bigcap_{i=1}^n T(y_i)$ and $\varphi_{\{x\}}(e_0)=\{x\}$, then T is a GFC-KKM mapping.

Proof.

(1) Suppose that the family $\{T(y):y\in Y\}$ has not the finite intersection property, then $\bigcap_{i=1}^n T(y_i)=\varnothing$. Since T is a GFC-KKM mapping, there exists a continuous mapping φ_N such that $\varphi_N(\Delta_k)\subset\bigcup_{j=1}^k T(y_{i_j})$ for any $\{y_{i_1},y_{i_2},...,y_{i_k}\}\in\{y_i,y_2,...,y_n\}$, especially, $\varphi_N(\Delta_n)\subset\bigcup_{i=1}^n T(y_i)$. Hence, $\varphi_N(\Delta_k)\subset\bigcup_{i=1}^n(X\cap T(y_i))$, because φ_N is continuous, then $\varphi_N(\Delta_n)$ is a compactly set, and since T is a set-valued mapping with compactly closed values, and hence, $\varphi_N(\Delta_n)\bigcap T(y_i)$ is closed in $\varphi_N(\Delta_n)$ for $\forall i\in\{1,2,...,n\}$, then $\varphi_N(\Delta_n)\setminus(\varphi_N(\Delta_n)\bigcap T(y_i))$ is open in $\varphi_N(\Delta_n)$. Then we have $\varphi_N(\Delta_n)=\bigcup_{i=1}^n(\varphi_N(\Delta_n)\setminus(\varphi_N(\Delta_n)\bigcap T(y_i)))$.

Hence, $\{\varphi_N(\Delta_n)\setminus(\varphi_N(\Delta_n)\bigcap T(y_i))\}_{i=1}^n$ is an open cover of $\varphi_N(\Delta_n)$. Let $\{\psi_i\}_{i=1}^n$ be the continuous partition of unity subordinate to the open covering, then we have that for each $i\in\{1,2,...,n\}$ and $x\in\varphi_N(\Delta_n)$,

$\psi_i(x)\neq 0\Leftrightarrow x\in\varphi_N(\Delta_n)\setminus(\varphi_N(\Delta_n)\bigcap T(y_i))$.

Define a mapping $\Psi:\varphi_N(\Delta_n)\to\Delta_n$ by $\Psi(x)=\sum_{i=1}^n\psi_i(x)e_i$, $\forall x\in\varphi_N(\Delta_n)$. Obviously, $\Psi\circ\varphi_N:\Delta_n\to\Delta_n$ is continuous. By the Brouwer fixed-point theorem, there exists a point $z_0\in\Delta_n$ such that $z_0=\Psi\circ\varphi_N(z_0)$. Let $z^*=\varphi_N(z_0)$, then we have $z^*=\varphi_N(z_0)=\varphi_N\circ\Psi\circ\varphi_N(z_0)=\varphi_N\circ\Psi(z^*)$.

and

$\Psi(z^*)=\sum_{i=1}^n\psi_i(z^*)e_i=\sum_{j\in\Delta_{J(z^*)}}\psi_j(z^*)e_j\in\Delta_{J(z^*)}$.

where $J(z^*)=\{j\in\{1,2,...,n\}:\psi_j(z^*)\neq 0\}$ and $\Delta_{J(z^*)}=co\{e_j:j\in J(z^*)\}$. We know that, $z^*\in\varphi_N(\Delta_n)\setminus(\varphi_N(\Delta_n)\bigcap T(y_j)),\forall j\in J(z^*)$, then $z^*\notin T(y_j)$. Since T is a GFC-KKM mapping, therefore, we have $z^*=\varphi_N(z_0)=\varphi_N\circ\Psi(z^*)\in\varphi_N(\Delta_{J(z^*)})\subset\bigcup_{j\in J(z^*)}T(y_j)$. Thus, there exists $j_0\in J(z^*)$ such that $z^*\in T(y_{j_0})$, this is contradictory. Therefore, the family $\{T(y):y\in Y\}$ has the finite intersection property.

(1) Suppose that the family $\{T(y):y\in Y\}$ has the finite intersection property, then for any $\{y_1,y_2,...,y_n\}\in\langle Y\rangle,\bigcap_{i=1}^n T(y_i)\neq\varnothing$. Let $x^*\in\bigcap_{i=1}^n T(y_i)$ and $x^*=x_i, i=1,2,\cdots,n$. Then for any $\{i_1,i_2,\cdots,i_k\}\subset\{1,2,\cdots,n\}$, we have $\{x^*\}=\{x_{i_1},x_{i_2},\cdots,x_{i_k}\}$. Thus, $\varphi_N(\Delta_k)=\varphi_{\{x^*\}}(e_0)=\{x^*\}\subset\bigcup_{j=1}^k T(y_{i_j})$.

Therefore, T is a GFC-KKM mapping.

Remark 2.1. Theorem 2.1 generalizes Theorem 2.1 of [1] and Theorem 1 of [5] from L-convex space to GFC-space.

Theorem 2.2. Let (X, Y, Φ) be a GFC-space, and $T: Y \to 2^X$ be a set-valued mapping with compactly open values.

1. If T is a GFC-KKM mapping, then the family $\{T(y): y \in Y\}$ has the finite intersection property.
2. If the family $\{T(y): y \in Y\}$ has the finite intersection property, there exists $x \in X$ such that $x \in \bigcap_{i=1}^{n} T(y_i)$ and $\varphi_{\{x\}}(e_0) = \{x\}$, then T is a GFC-KKM mapping.

Proof.

(1) Suppose that the family $\{T(y): y \in Y\}$ has not the finite intersection property, then $\bigcap_{i=1}^{n} T(y_i) = \varnothing$, then we have $\varphi_N(\Delta_n) \cap (\bigcap_{i=1}^{n} T(y_i)) = \varnothing$. It follows that

$$\varphi_N(\Delta_n) = \bigcup_{i=1}^{n} (\varphi_N(\Delta_n) \setminus (T(y_i) \cap \varphi_N(\Delta_n))).$$

Since φ_N is continuous, then $\varphi_N(\Delta_n)$ is a compactly set, and since T is a set-valued mapping with compactly open values, and hence, $\varphi_N(\Delta_n) \bigcap T(y_i)$ is open in $\varphi_N(\Delta_n)$ for $\forall i \in \{1, 2, ..., n\}$. For each $z \in \Delta_n$, let $I(z) = \{i \in \{1, 2, ..., n\} : \varphi_N(z) \notin T(y_i)\}$ and $S(z) = co(\{e_i : i \in I(z)\})$. If for some $z \in \Delta_n$, $I(z) = \varnothing$. Then we have $\varphi_N(z) \in T(y_i)$ for all $i \in \{1, 2, ..., n\}$ which contradicts the assumption $\varphi_N(\Delta_n) \cap (\bigcap_{i=1}^{n} T(y_i)) = \varnothing$.

Therefore we can assume that $I(z) \neq \varnothing$ for each $z \in \Delta_n$ and hence $S(z)$ is a nonempty compact convex subset of Δ_n. Since $\bigcup_{i \notin I(z)} (\varphi_N(\Delta_n) \setminus (T(y_i) \cap \varphi_N(\Delta_n)))$ is closed in $\varphi_N(\Delta_n)$, we have that $U = \Delta_n \setminus \varphi_n^{-1}(\bigcup_{i \notin I(z)} (\varphi_N(\Delta_n) \setminus (T(y_i) \cap \varphi_N(\Delta_n))))$ is an open neighborhood of z in Δ_n. For $\forall z' \in U$, we have $\varphi_N(z') \in T(y_i), i \notin I(z)$, and hence, $I(z') \subset I(z)$, It follows that $S(z') \subset S(z)$.

This shows that $S: \Delta_n \to 2^{\Delta_n}$ is an upper semicontinuous set-valued mapping with nonempty compact convex values. By the Kakutani fixed point theorem, there exists $z_0 \in \Delta_n$ such that $z_0 \in S(z_0)$. Note that T is a GFC-KKM mapping, then we have $\varphi_N(z_0) \subset \bigcup_{i \in I(z_0)} T(y_i)$, Hence there exists a $i_0 \in I(z_0)$ such that $\varphi_N(z_0) \in T(y_{i_0})$. By the definition of $I(z_0)$, we have $\varphi_N(z_0) \notin T(y_i)$ for each $i \in I(z_0)$, which is a contradiction. Therefore the family $\{T(y): y \in Y\}$ has the finite intersection property.

(2) The proof is the same as Theorem 2.1.

Remark 2.2. Theorem 2.2 generalizes Theorem 2.3 of [1] from L-convex space to GFC-space.

Theorem 2.3. Let (X, Y, Φ) be a GFC-space, $S: Y \to 2^X$ and $T: X \to 2^X$ are two multivalued mappings such that:

(i) $S(y)$ is closed for all $y \in Y$;

(ii) For any $x \in X$, $N = \{y_1, y_2, \cdots, y_n\} \in Y$ and $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \{y_1, y_2, ..., y_n\}$, if $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \langle Y \setminus S^{-1}(x) \rangle$ can deduce $\varphi_N(\Delta_k) \subset X \setminus T^{-1}(x)$;

(iii) $x \in T(x)$ for all $x \in X$;

Then the family $\{S(y): y \in Y\}$ has the finite intersection property.

Proof. By Theorem 2.1, it is only necessary to prove that $S: Y \to 2^X$ is a GFC-KKM mapping. Suppose $S: Y \to 2^X$ is not a GFC-KKM mapping, there exists $N = \{y_1, y_2, \cdots, y_n\} \in Y$ and $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \{y_1, y_2, ..., y_n\}$ such that $\varphi_N(\Delta_k) \not\subset \bigcup_{j=1}^{k} S(y_{i_j})$. There exists $x^* \in \varphi_N(\Delta_k)$ such that $x^* \notin \bigcup_{j=1}^{k} S(y_{i_j})$, hence, for each $j \in \{1, 2, ..., k\}$, we have $y_{i_j} \notin S^{-1}(x^*)$. Therefore $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \langle Y \setminus S^{-1}(x^*) \rangle$. By condition(ii), $x^* \in \varphi_N(\Delta_k) \subset X \setminus T^{-1}(x^*)$, that is, $x^* \notin T(x^*)$, this contradicts condition (iii). Therefore the family $\{S(y): y \in Y\}$ has the finite intersection property.

Remark 2.3. Theorem 2.3 generalizes Theorem 2.2 in [10] from FC-space to GFC-space.

Theorem 2.4. Let (X,Y,Φ) be a GFC-space, $S:Y\to 2^X$ and $T:X\to 2^X$ are two multivalued mappings such that:
(i) $S(y)$ is open for all $y\in Y$;
(ii) For any $x\in X$, $N=\{y_1,y_2,\cdots,y_n\}\in Y$ and $\{y_{i_1},y_{i_2},...,y_{i_k}\}\in\{y_1,y_2,...,y_n\}$, if $\{y_{i_1},y_{i_2},...,y_{i_k}\}\in\langle Y\setminus S^{-1}(x)\rangle$ can deduce $\varphi_N(\Delta_k)\subset X\setminus T^{-1}(x)$;
(iii) $x\in T(x)$ for all $x\in X$;
Then the family $\{S(y):y\in Y\}$ has the finite intersection property.

Proof. From the proof part of Theorem 2.3, it is easy to know that the conclusion is valid.

Remark 2.4. Theorem 2.4 proves that Theorem 2.3 also holds under the condition of compactly opening.

3. Applications

Theorem3.1. Let (X,Y,Φ) be a GFC-space, and $T:Y\to 2^X$ be a set-valued mapping with compactly closed values. If T is a GFC-KKM mapping, then we have $\varphi_N(\Delta_n)\cap(\bigcap_{i=1}^n T(y_i))\neq\varnothing$.

Proof. Since T is a GFC-KKM mapping, then there is a continuous mapping φ_N, then for any $\{y_{i_1},y_{i_2},...,y_{i_k}\}\in\{y_1,y_2,...,y_n\}$, we have $\varphi_N(\Delta_k)\subset\bigcup_{j=1}^k T(y_{i_j})$. Let $A_i=\varphi_N^{-1}(\varphi_N(\Delta_k)\cap T(y_i))$ for $\forall z\in\Delta_k$, then we have $\varphi_N(z)\in\varphi_N(\Delta_k)\subset\varphi_N(\Delta_n)$ and $\varphi_N(z)\in\varphi_N(\Delta_k)\subset\bigcup_{j=1}^k T(y_{i_j})$, hence, there exists $j_0=\{1,2,\cdots,k\}$ such that $\varphi_N(z)\in T(y_{i_{j0}})$, then $\varphi_N(z)\in\varphi_N(\Delta_n)\cap T(y_{i_{j0}})$, therefore, we have $z\in\varphi_N^{-1}(\varphi_N(\Delta_n)\cap T(y_{i_{j0}}))$, thus, $\Delta_k\subset\bigcup_{j=1}^k\varphi_N^{-1}(\varphi_N(\Delta_n)\cap T(y_{i_j}))=\bigcup_{j=1}^k A_{i_j}$.
Since $\varphi_N(\Delta_n)$ is compact and T is a GFC-KKM mapping, and hence, $\varphi_N(\Delta_n)\cap T(y_i)$ is a closed set, by the continuity of φ_N, then it is clear that $A_i=\varphi_N^{-1}(\varphi_N(\Delta_k)\cap T(y_i))$ is a closed set for $\forall i\in\{1,2,\cdots,n\}$. From the Classical KKM Theorem, $\bigcap_{i=1}^n A_i\neq\varnothing$, therefore, $\varphi_N(\Delta_n)\cap(\bigcap_{i=1}^n T(y_i))=\bigcap_{i=1}^n\varphi_N(A_i)\neq\varnothing$, that is, $\varphi_N(\Delta_n)\cap(\bigcap_{i=1}^n T(y_i))\neq\varnothing$.

Remark 3.1. Theorem 3.1 generalizes Theorem 2.1 in [1] from L-convex space to GFC-space.

Theorem 3.2. Let (X,Y,Φ) be a GFC-space, and $T:Y\to 2^X\setminus\{\varnothing\}$ be a set-valued mapping with compactly closed values, there exists $M\in\langle Y\rangle$ and $N=\{x_1,x_2,\cdots,x_n\}\in\langle X\rangle$ such that $\varphi_N(\Delta_n)$ is a compact set, and $\bigcap_{y\in M}T(y)\subset\varphi_N(\Delta_n)$, then $\varphi_N(\Delta_n)\cap(\bigcap_{y\in Y}T(y))\neq\varnothing$.

Proof. Let $G(y)=T(y)\cap(\bigcap_{y\in M}T(y))$, $\forall y\in Y$, then $G(y)\subset\bigcap_{y\in M}T(y)\subset\varphi_N(\Delta_n)$, and hence, $G(y)=G(y)\cap\varphi_N(\Delta_n)=\bigcap_{z\in\{y\}\cap M}(T(z)\cap\varphi_N(\Delta_n))$. By theorem 3.1, $\varphi_N(\Delta_n)\cap(\bigcap_{i=1}^n T(y_i))\neq\varnothing$, then we have $\bigcap_{i=1}^n G(y_i)\neq\varnothing$. Then the family $\{G(y):y\in Y\}$ has the finite intersection property, thus, $\bigcap_{y\in Y}G(y)\neq\varnothing$, then $\varphi_N(\Delta_n)\cap(\bigcap_{y\in Y}T(y))\neq\varnothing$.

Remark 3.2. Theorem 3.2 generalizes Theorem 3 in [5] from FC-space to GFC-space, and T is a GFC-KKM mapping instead of S-G-FC-KKM mapping.

Theorem 3.3. Let (X,Y,Φ) be a GFC-space, $\{A_i\}_{i=1}^n$ be a family of compactly closed subsets of X such that $\bigcup_{i=1}^n A_i=X$ and $y_1,y_2,...,y_n$ be n points of Y. Then for any n points $x_1,x_2,\cdots,x_n$ in X there exists $\{x_{i_1},x_{i_2},\cdots,x_{i_k}\}\subset\{x_1,x_2,\cdots,x_n\}$ such that $\varphi_N(\Delta_n)\cap(\bigcap_{j=1}^k A_{i_j})\neq\varnothing$.

Proof. Let $Y_1=\{y_1,y_2,\cdots y_n\}$, for any $A=\{x_1,x_2,\cdots,x_n\}\subset X$, define a set-valued mapping $G:Y_1\to 2^X$ by $G(y_i)=X\setminus A_i$ for each $i=1,2,...,n$. Since each A_i is compactly closed, then G is a set-valued mapping with compactly open values.

Suppose that the conclusion is false, then for any $\{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset \{x_1, x_2, \cdots, x_n\}$, we have $\varphi_N(\Delta_n) \cap (\bigcap_{j=1}^k A_{i_j}) = \varnothing$ and so $\varphi_N(\Delta_n) \subset \bigcup_{j=1}^k (X \setminus A_{i_j}) = \bigcup_{j=1}^k G(y_{i_j})$, i.e., $G: Y_1 \to 2^X$ is a GFC-KKM mapping. By theorem 2.2, $\bigcap_{i=1}^n G(y_i) \neq \varnothing$. It follows that $\bigcup_{i=1}^n A_i \neq X$ which contradicts the assumption $\bigcup_{i=1}^n A_i = X$. Hence, the conclusion holds.

Remark 3.3. Theorem 3.3 generalizes Theorem 3.1 in [3] from L-convex space to GFC-space.

Theorem3.4. Let (X, Y, Φ) be a GFC-space, $\{A_i\}_{i=1}^n$ be a family of compactly open subsets of X such that $\bigcup_{i=1}^n A_i = X$ and $y_1, y_2, ..., y_n$ be n points of Y. Then for any n points $x_1, x_2, \cdots, x_n$ in X there exists $\{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset \{x_1, x_2, \cdots, x_n\}$ such that $\varphi_N(\Delta_n) \cap (\bigcap_{j=1}^k A_{i_j}) \neq \varnothing$.

Proof. Let $Y_1 = \{y_1, y_2, \cdots y_n\}$, for any $A = \{x_1, x_2, \cdots, x_n\} \subset X$, define a set-valued mapping $G: Y_1 \to 2^X$ by $G(y_i) = X \setminus A_i$ for each $i = 1, 2, ..., n$. Since each A_i is compactly open, then G is a set-valued mapping with compactly closed values. Suppose that the conclusion is false, then for any $\{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset \{x_1, x_2, \cdots, x_n\}$, we have $\varphi_N(\Delta_n) \cap (\bigcap_{j=1}^k A_{i_j}) = \varnothing$ and so $\varphi_N(\Delta_n) \subset \bigcup_{j=1}^k (X \setminus A_{i_j}) = \bigcup_{j=1}^k G(y_{i_j})$, i.e., $G: Y_1 \to 2^X$ is a GFC-KKM mapping. By theorem 2.2, $\bigcap_{i=1}^n G(y_i) \neq \varnothing$. It follows that $\bigcup_{i=1}^n A_i \neq X$ which contradicts the assumption $\bigcup_{i=1}^n A_i = X$. Hence, the conclusion holds.

Remark 3.4. Theorem 3.4 generalizes Theorem 3.2 in [3] from L-convex space to GFC-space.

Theorem 3.5. Let (X, Y, Φ) be a GFC-space, and $T: Y \to 2^X$ be a set-valued mapping with compactly closed values. Suppose that there exists n points $y_1, y_2, ..., y_n$ in Y such that $X = \bigcup_{i=1}^n T(y_i)$, and for $\forall x \in X$, $T^{-1}(x) = \{y \in Y : x \in T(y)\}$ is GFC-convex. Then T has a fixed point in X.

Proof. By theorem 3.3, there exists $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \{y_1, y_2, ..., y_n\}$ such that $\varphi_N(\Delta_n) \cap (\bigcap_{j=1}^k T(y_{i_j})) \neq \varnothing$. Let $x* \subset \varphi_N(\Delta_n) \cap (\bigcap_{j=1}^k T(y_{i_j}))$, then $y_{i_j} \in T^{-1}(x*)$ for all $j = 1, 2, \cdots k$. Since $T^{-1}(x*)$ is GFC-convex, we have $x* \in \varphi_N(\Delta_k) \subset T^{-1}(x*)$, i.e., T has a fixed point in X.

Remark 3.5. Theorem 3.5 generalizes Theorem 3.3 in [1] from L-convex space to GFC-space.

Theorem 3.6. Let (X, Y, Φ) be a GFC-space, and $T: Y \to 2^X$ be a set-valued mapping with compactly open values. Suppose that there exists n points $y_1, y_2, ..., y_n$ in Y such that $X = \bigcup_{i=1}^n T(y_i)$, and for $\forall x \in X$, $T^{-1}(x) = \{y \in Y : x \in T(y)\}$ is GFC-convex. Then T has a fixed point in X.

Proof. By theorem 3.4 and theorem 3.5, it is easy to prove that the conclusion holds.

Remark 3.6. Theorem 3.6 generalizes Theorem 3.4 in [1] from L-convex space to GFC-space.

Theorem 3.7. Let (X, Y, Φ) be a GFC-space, $S: Y \to 2^X$ and $T: X \to 2^X$ are two multivalued mappings such that:

(i) $S(y)$ is compactly closed for all $y \in Y$;

(ii) For any $x \in X$, $N = \{y_1, y_2, \cdots, y_n\} \in Y$ and $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \{y_1, y_2, ..., y_n\}$, if $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \langle S^{-1}(x) \rangle$ can deduce $\varphi_N(\Delta_k) \subset T^{-1}(x)$;

(iii) There exists $N = \{y_1, y_2, \cdots, y_n\} \in Y$ such that $S^{-1}(x) \cap N \neq \varnothing$ for any $x \in X$.

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. Define two set-valued mapping $A: Y \to 2^X$ by $A(y) = X \setminus S(y)$ for $\forall y \in Y$ and $B: X \to 2^X$ by $B(x) = X \setminus T(x)$ for $\forall x \in X$. Then $A(y)$ is compactly open for $\forall y \in Y$, and for $\forall x \in X$, we have

$Y \setminus A^{-1}(x) = S^{-1}(x)$ and $X \setminus B^{-1}(x) = T^{-1}(x)$. By condition(iii), $(Y \setminus A^{-1}(x)) \cap N \neq \varnothing, \forall x \in X$. Take $y^* \in (Y \setminus A^{-1}(x)) \cap N$, then we have $x \in X \setminus A(y^*)$, that is, $X = \bigcup_{y\in N}(X \setminus A(y))$, i.e., $\bigcap_{y\in N} A(y) = \varnothing$. By theorem 2.2, T is not a GFC-KKM mapping, then there exists $N = \{y_1, y_2, \cdots, y_n\} \in Y$ and $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \{y_1, y_2, ..., y_n\}$ such that $\varphi_N(\Delta_k) \not\subset \bigcup_{j=1}^{k} A(y_{i_j})$, that is, there exists $x^* \in \varphi_N(\Delta_k)$ such that $x^* \notin \bigcup_{j=1}^{k} A(y_{i_j})$, then $y_{i_j} \notin A^{-1}(x^*)$, $\forall j \in \{1, 2, \cdots, k\}$, i.e., $\{y_{i_1}, y_{i_2}, ..., y_{i_k}\} \in \langle Y \setminus A^{-1}(x^*)\rangle$.

By condition(ii), $\varphi_N(\Delta_k) \subset T^{-1}(x)$, that is, $\varphi_N(\Delta_k) \subset X \setminus B^{-1}(x)$. Therefore $x^* \in \varphi_N(\Delta_k) \subset X \setminus B^{-1}(x)$, that is, $x^* \notin B(x^*) = X \setminus T(x^*)$. Hence, the conclusion holds.

Remark 3.7. Theorem 3.7 generalizes Theorem 2.3 in [10] from FC- space to GFC-space.

Theorem3.8. Let (X, D, Φ) be a GFC-space, $f : D \times X \to R$ and $g : X \times X \to R$ two functions, let $v = \sup_{x\in X} \inf_{y\in X}(x, y)$. If

(i) $\{y \in X : f(x, y) < \lambda\}$ is compactly open for any $x \in D$;

(ii) $\{x \in X : g(x, y) > \lambda\}$ is GFC-convex for any $y \in X$;

(iii) $f(x, y) < g(x, y)$ for any $(x, y) \in C = \{(x, y) \in D \times X : f(x, y) > v\}$;

(iv) $g(x, x) \leq v$ for any $x \in X$.

Then $\sup_{N\in\langle D\rangle} \inf_{y\in X} \sup_{x\in N} f(x, y) \leq \sup_{x\in X} \inf_{y\in X}(x, y)$.

Proof. If $v = +\infty$, then this Theorem is obviously true, hence we may assume $v < +\infty$. For any fixed $\lambda \in R$ such that $\lambda > v$, define two maps as follows:

$S : D \to X, S(x) = \{y \in X : f(x, y) < \lambda\}, \forall x \in D;$

$T : X \to X, T(x) = \{y \in X : g(x, y) \leq \lambda\}, \forall x \in X;$

Then by condition(i), $S(x)$ is compactly open for any $x \in D$, if $N \in \langle D \setminus S^{-1}(y)\rangle$ for any $y \in X$, then $N \in \langle D\rangle$, and $y \notin S(x)$ for any $x \in N$, that is, $f(x, y) \geq \lambda$. i.e., by condition(iii), $g(x, y) > f(x, y) \geq \lambda > v$ for $\forall x \in N$, then we have $N \subset \{x \in D : g(x, y) > \lambda\} \subset \{x \in: g(x, y) > \lambda\}$.

Then by condition(ii), $\varphi_N(\Delta_k) \subset \{x \in X : g(x, y) > \lambda\} = X \setminus T^{-1}(y)$; by condition(iv), $g(x, x) \leq v < \lambda$ for any $x \in X$, therefore $x \in T(x)$. Hence S and T satisfy all the conditions of Theorem 2.3, then the family $\{S(x) : x \in D\}$ has the finite intersection property, therefore $\bigcap_{x\in N} S(x) \neq \varnothing$ for any $N \in \langle D\rangle$. Take any $y_0 \in \bigcap_{x\in N} S(x)$, then $f(x, y_0) < \lambda, \forall x \in N$, i.e., $\inf_{y\in X} \sup_{x\in N} f(x, y) \leq \lambda$. Since N is arbitrary, then we have $\sup_{N\in\langle D\rangle} \inf_{y\in X} \sup_{x\in N} f(x, y) \leq \lambda$. And λ is arbitrary, let $\lambda \to v$, hence $\sup_{N\in\langle D\rangle} \inf_{y\in X} \sup_{x\in N} f(x, y) \leq v = \sup_{x\in X} \inf_{y\in X}(x, y)$.

Remark 3.8. Theorem 3.8 generalizes Theorem 2.4 in [10] from FC- space to GFC-space.

Theorem 3.9. Let (X, D, Φ) be a GFC-space, $f : D \times X \to R$ and $g : X \times X \to R$ two functions, let $v = \sup_{x\in X} \inf_{y\in X}(x, y)$. If

(i) $\{y \in X : f(x, y) < \lambda\}$ is compactly closed for any $x \in D$;

(ii) $\{x \in X : g(x, y) > \lambda\}$ is GFC-convex for any $y \in X$;

(iii) $f(x, y) < g(x, y)$ for any $(x, y) \in C = \{(x, y) \in D \times X : f(x, y) > v\}$;

(iv) $g(x, x) \leq v$ for any $x \in X$.

Then $\sup_{N\in\langle D\rangle} \inf_{y\in X} \sup_{x\in N} f(x, y) \leq \sup_{x\in X} \inf_{y\in X}(x, y)$.

Proof. From the proof part of Theorem 2.3, it is easy to know that the conclusion is valid.

Remark 3.9. Theorem 3.9 proves that Theorem 3.8 also holds under the condition of compactly closed.

REFERENCES

1. Ding, X. P. (2002). Generalized L-KKM type theorems in L-convex spaces with applications [J]. *Computers & Mathematics with Applications*, 43(10-11): 1249-1256.
2. Khanh, P. Q., Quan, N. H., & Yao, J. C. (2009). Generalized KKM-type theorems in GFC-spaces and applications [J]. *Nonlinear Analysis: Theory, Methods & Applications*, 71(3-4): 1227-1234.
3. Fang, M., & Ding, X. P. (2003). Generalized L-R-KKM Type Theorems and Applications in L-convex spaces [J]. *Journal of Sichuan Normal University*, (5), 461-463.
4. Cai-Yun, J., & Cao-Zong C. (2005). S-G-L-KKM theorems in L-convex space and their applications to minimax inequalities [J]. *Computers and Mathematics with Applications*, 50(1).
5. Wang, T. (2007). G-FC-KKM Type Theorems in FC-space [J]. *Journal of Chongqing Jiaotong University*, (5):158-160.
6. George Xian-Zhi, Y. (1999). The Characterization of Generalized Metric KKM Mappings with Open Values in Hyperconvex Metric Spaces and Some Applications [J]. *Journal of Mathematical Analysis and Applications*, 235(1).
7. Wen Kai-ting. (2018). Intersection Theorems in GFC-metric Spaces with Applications to System of General Quasiequilibrium Problems [J]. *Journal of Guizhou University of Engineering Science*, 36(03):55-62.
8. Kai-Ting, W., & He-Rui, L. (2015). A GFC-KKM Theorem in GFC-spaces with the Application to Fixed Points [J]. *Journal of Sichuan Normal University*, 38(03):386-390.
9. Wen Kai-ting. (2010). R-KKM Theorems in FC-Metric spaces with Application to Variational inequalities and fixed point [J]. *ACTA Analysis Functionalis Applicata*, 12(3), 266-273.
10. Wang, B., & Liu, H. (2013). Fixed point theorem and supinfsup inequalities in FC-space without compactness [J]. *Journal of NeiJiang Normal University*, 28(10):4-7.
11. Horvath, C. (1987). Some reults on multivalued mappings and inequalities without convexity [J]. *Pure and Appl Math Series*, 106: 99-106.
12. Verma, R. U. (1999). G-H-KKM type theorems and their applications to a new class of minimax inequalities [J]. Computers and Mathematics with Applications, 37(8).
13. Ben-El-Mechaiekh, H., Chebbi, S., & Florenzano, M. (1998). Abstract convexity and fixed points [J]. *Journal of Mathematical Analysis and Applications*, 222(1): 138-150.
14. Ding, X. P. (2005). Maximal element theorems in product FC-spaces and generalized games [J]. *Journal of Mathematical Analysis and Applications*, 305(1): 29-42.
15. Xie Ping Ding., & Lei Wang. (2007). Fixed points, minimax inequalities and equilibria of noncompact abstract economies in FC –spaces [J]. *Nonlinear Analysis*, 69(2).
16. Sehie, P., & Hoonjoo, K. (1997). Foundations of the KKM Theory on Generalized Convex Spaces [J]. *Journal of Mathematical Analysis and Applications*, 209(2).
17. Ding, X. P. (2002). Generalized G-KKM theorems in generalized convex spaces and their applications [J]. *Journal of Mathematical Analysis and Applications*, 266(1): 21-37.
18. Park, S. H. (2000). Fixed points of better admissible maps on generalized convex spaces [J]. *Journal of the Korean Mathematical Society*, 37(6): 885-899.
19. Ding, X. P. (2006). Generalized KKM type theorems in FC-spaces with applications (I) [J]. *Journal of Global Optimization*, 36(4): 581-596.
20. Ding, X. P. (2007). Generalized KKM type theorems in FC-spaces with applications (II) [J]. *Journal of Global Optimization*, 38(3): 367-385.
21. Li Hong-mei. (2002). On Some Versions of KKM Principle in G-H-space [J]. *Journal of Sichuan Normal University*, 25(2): 115-117.
22. Lassonde, M. (1983). On the use of KKM multifunctions in fixed point theory and related topics [J]. *Journal of Mathematical Analysis and Applications*, 97(1): 151-201.
23. Ding, X. P., Liou Y. C., & Yao, J. C. (2005). Generalized R-KKM type theorems in topological spaces with applications [J]. *Applied mathematics letters*, 18(12): 1345-1350.
24. Ding, X. P., & Ding, T. M. (2007). KKM type theorems and generalized vector equilibrium problems in noncompact FC-spaces [J]. *Journal of Mathematical Analysis and Applications*, 331(2): 1230-1245.
25. Verma, R. U. (1999). Some results on R-KKM mappings and R-KKM selections and theirapplications [J]. *Journal of Mathematical Analysis and Applications*, 232(2), 428-433.